state $|\alpha^i>$) by $|i>$;

$$|i>\equiv|\alpha^i>\prod_{j(\neq i)}|\beta^j>. \quad (22)$$

The Hamiltonian is assumed to be given by

$$\hat{H}=\sum_{<ij>}J_{ij}\hat{H}_{ij}=\sum_{<ij>}J_{ij}\left(|i><j|+|j><i|\right), \quad (23)$$

where $\hat{H}_{ij}$ expresses a two-spin operator

$$\hat{H}_{ij} \equiv \frac{1}{2}(\sigma^i_x\sigma^j_x+\sigma^i_y\sigma^j_y)=\sigma^i_+\sigma^j_-+\sigma^i_-\sigma^j_+ = |i><j|+|j><i|, \quad (24)$$

and $\sigma^i_\pm$ is defined by

$$\sigma^i_\pm\equiv\frac{1}{2}(\sigma^i_x\pm i\sigma^i_y). \quad (25)$$

For this system, we have

$$[\sum_{i=1}^N\sigma^i_z,\ \hat{H}]=0 \quad (26)$$

or the z-component of the total spin $\sum_{i=1}^N\sigma^i_z$ is conserved. The state space of this Hamiltonian spanned by the basis

$$\{\prod_{i=1}^N|\alpha^i \text{ or } \beta^i>\} \quad (27)$$

is thus decomposed into subspaces according to the value of the z-component of the total spin and $|\Psi(t)>$ remains in the same subspace as $|\Psi(t=0)>$.

In what follows, we consider the transfer of information and the formation of entanglement starting from the initial state where only one spin at the end is excited. Therefore, *When we start from the state with only one spin is excited, the state of the system stays within the subspace spanned by* $\{|i>\}_{i=1,\dots,N}$.

3. SOME MODELS AND TRANSFER EFFICIENCY

3.1 Models

In this Section, we consider three kinds of spin chains, Models I, II, and III:

Model I[2]

$$J_{1,2} = J_{N-1,N}=a, \quad (31)$$

$$J_{i,i+1}=J_{i+1,i} = 1,\quad i=2,\dots,N-2. \quad (32)$$

Here $a<1$ is an adjustable parameter and, when appropriate value of a is chosen, the efficiency is shown to be comparable to unity.

Model II[3, 4]

$$J_{i,i+1}=J_{i+1,i}=[i(N-i)]^{1/2}\quad i=1,\dots,N-1. \quad (33)$$

This chain has the property of the perfect transfer or the efficiency is unity.

Model III[7]

$$J_{1,2} = J_{N-1,N}=a, \quad (34)$$

$$J_{i,i+1}=J_{i+1,i} = [i(N-i)]^{1/2}\quad i=2,\dots,N-2 \quad (35)$$

The element of Model I is added to Model II.

The initial state is given by

$$|\Psi(t=0)>=|1> \quad (36)$$

and the time development of the spin chain is expressed as

$$|\Psi(t)>=\sum_j a_j(t)|j>. \quad (37)$$

The efficiency of transfer is measured by the maximum of the expectation value of spin flip at $j=N$ or

$$P_{N,max}(t)=\max_{0\le t'\le t}\{P_N(t')\}, \quad (38)$$

where

$$P_N(t)=|<N|\Psi(t)>|^2=|a_N(t)|^2. \quad (39)$$

This value is also called fidelity.

We here consider the effects of noises which are expressed as fluctuations in the coupling between spins. The Hamiltonian is then expressed as

$$\hat{H}+\hat{H}', \quad (310)$$

where $\hat{H}$ is given by Models I, II, and III and $\hat{H}'$ expresses noises in the coupling constant. For the Hamiltonian to be hermitian, the matrix elements of $\hat{H}'$ is assumed to be real and symmetric.

In simulations, we introduce random numbers $-0.5<\delta_i<0.5$ and modify the matrix element as

$$(H+H')_{ij}=(H+H')_{ji}=H_{ij}(1+\epsilon\delta_i). \quad (311)$$

Here ϵ is the strength of perturbations.

3.2 Results of Simulations and Discussions

We have performed simulations of the time development for Model I, II, and III with $N\le 500$. We note that, since the performance is closely related to the structure of the eigenvalue around 0, the parity of the Hamiltonian matrix has a significant effect. We find that generally the case of even parity ($N=$ even) gives better results. We here present the results for $N=32$.

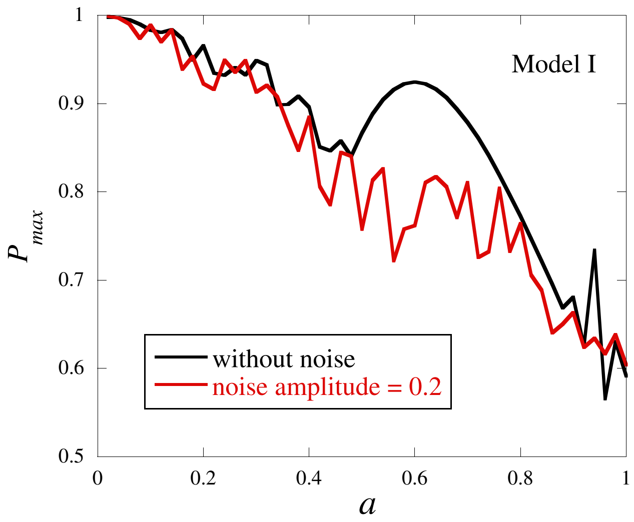

Fig. 1 Maximum probability vs. value of a in Model I without noise and with noise of relative amplitude 0.2.

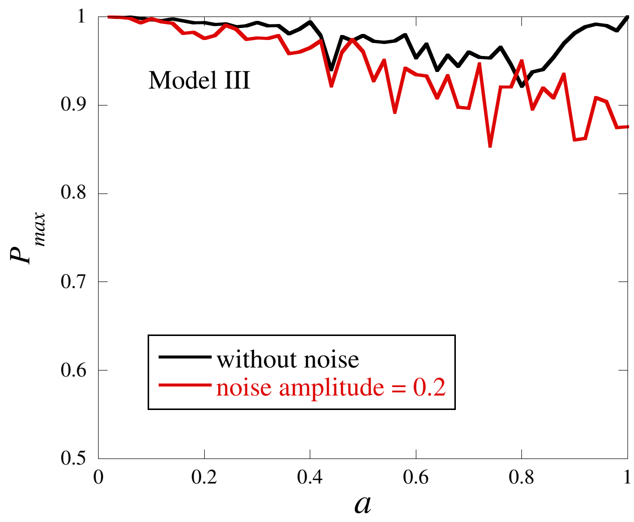

Fig. 2 Maximum probability vs. value of a in Model III without noise and with noise of relative amplitude 0.2.

In Model I, the fidelity never attains unity. As shown in Ref.[2], the maximum is controlled by a and is obtained when $a \sim 0.6$ as is shown in Fig.1. The effect of noise is also shown in Fig.1.

In Model II, $P_{N,max}(t)$ take on exactly unity when t is sufficiently large (perfect transfer). This is due to the structure of eigenvalues for this Hamiltonian: They are exactly of equal spacing. When we expand the initial state into eigenstates, the expansion coefficient has a broad spectrum. Since the perfect transfer is achieved by the superposition of eigenstates within the broad spectrum, the noise in the tuned coupling constants has a large effect on the transfer[5].

In Model III, the coupling constant a on both sides is regarded as a control parameter. The structure of eigenstates for small values of a and without noises is analyzed in Appendix. We observe that the value of $P_{N,max}$ is close to unity as shown in Fig.2 and, at the same time, the effect of noise is sufficiently small as also shown in Fig.2. *We thus propose to use Model III for quantum information transfer.*

The effect of noise is small when the initial state has a narrow spectrum. In this case, the maximum of $P_N(t)$ is mainly determined by the beat of two eigenfrequencies and it may not be influenced strongly by the change of two eigenfequencies due to noise. If the maximum is determined by superposition of many eigenfrequencies, this 'collective' maximum may be strongly influenced by changes in component frequencies and with noises, it may be difficult to attain the maximum which is possible without noises. In Figs.3(a) and (b), the spectrum of the initial state is shown for various values of a in Models I and III. We observe that the spectrum becomes narrow with the decrease of the coupling at the ends.

In addition, when a is sufficiently small, the inner part of the Hamiltonian in Model III of $(N-2)^2$ dimensions has the equal spacing structure of eigenvalues, as shown in Appendix. The components of the initial state transferred to $(N-2)$-dimensional space, may thus have the property of perfect transfer. This is expected to give an advantages to Model III compared with Model I.

4. BRANCHED CHAIN AND ENTANGLEMENT CREATION

4.1 Creation of Entanglement

Let us consider the system with the Hamiltonian

$$\hat{H} = J_1|1><2| + J_2|2><3| + J_2|2><4| + h.c., \quad (41)$$

where $h.c.$ is the hermitian conjugate. This system is symmetric with respect to the exchange $3 \longleftrightarrow 4$. Therefore, (a) the parity with respect to the exchange $3 \longleftrightarrow 4$ is conserved, and (b) the eigenstate of this Hamiltonian has even or odd parity.

When we define $|3 \pm 4>$ by

$$|3 \pm 4> \equiv \frac{1}{2^{1/2}}(|3> \pm |4>), \quad (42)$$

and take $\{|1>, |2>, |3 \pm 4>\}$ in stead of $\{|1>, |2>, |3>, |4>\}$ as the basis, the space spanned by $\{|1>, |2>, |3>, |4>\}$ is decomposed into the subspaces with the even parity $\{|1>, |2>, |3+4>\}$ and the odd parity $\{|3-4>\}$. When we start from the initial condition $|\Psi(t=0)> = |1>$ of even parity, $|\Psi(t)>$ is in the 3-dimensional subspace spanned by $\{|1>, |2>, |3+4>\}$.

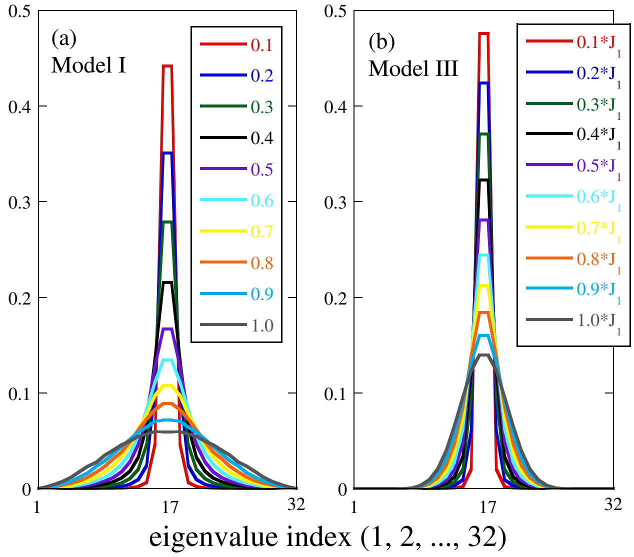

Fig. 3 Dependence of initial state spectrum on a in Model I (a) and Model III (b).

In this 3-dimensional subspace, we can rewrite the Hamiltonian into the form of linear chain

$$J_1|1><2| + 2^{1/2}J_2|2><3+4| + h.c. \quad (43)$$

and apply known properties of linear spin chains. Note that the coupling at the branch point has to be multiplied by $2^{1/2}$.

In the case where branches have longer chains such as

$$\begin{aligned}\hat{H} &= J_1|1><2| + J_2|2><3| + J_3|3><5| \\ &+J_2|2><4| + J_3|4><6| + h.c., \quad (44)\end{aligned}$$

we have the equivalent Hamiltonian

$$\begin{aligned}&J_1|1><2| + 2^{1/2}J_2|2><3+4| \\ &+J_3|3+4><5+6| + h.c., \quad (45)\end{aligned}$$

as far as the states starting from the same initial states are concerned.

4.2 Effect of Asymmetry

The creation of entanglement is closely related to the symmetry of the spin chain. Here we analyze the effect of asymmetry. Let us assume that the Hamiltonian is modified into the form

$$\begin{aligned}\hat{H} &= J_1|1><2| + (J_2+\delta)|2><3| \\ &+(J_2-\delta)|2><4| + h.c. \quad (46)\end{aligned}$$

by the effect of asymmetry δ.

The eigenvalues the Hamiltonian matrix are now given by

$$\lambda_\pm = \pm(J_1^2 + 2J_2^2 + 2\delta^2)^{1/2}, \quad (47)$$

$$\lambda = 0 \ \ \text{(doubly degenerate)}, \quad (48)$$

where the eigenvalue 0 is two-fold degeneracy. When expressed by the basis $\{|1>, |2>, |3+4>, |3-4>\}$, corresponding normalized eigenstates are given by

$$(J_1,\ \lambda_\pm,\ 2^{1/2}J_2,\ 2^{1/2}\delta)/2^{1/2}|\lambda_\pm|, \quad (49)$$

$$(2^{1/2}J_2,\ 0,\ -j_1,\ 0)/(J_1^2+2J_2^2)^{1/2}, \quad (410)$$

$$\begin{aligned}&(-2^{1/2}\delta,\ 0,\ -2\delta J_2,\ J_1^2+2J_2^2) \\ &/|\lambda_\pm|(J_1^2+2J_2^2)^{1/2}. \quad (411)\end{aligned}$$

Here orthogonalized eigenstates are taken in the degenerate space with the eigenvalue 0.

The creation of entanglement is given by the squared amplitude $|<3+4|\Psi(t)>|^2$. Since the time evolution of the state is given by

$$|\Psi(t)> = \sum_{1=1}^{4} e^{i\lambda_i t}|\lambda_i><\lambda_i|\Psi(t=0)> \quad (412)$$

and $|\Psi(t=0)> = |1>$, we have

$$\begin{aligned}|<3+4|\Psi(t)>|^2 &= \sum_{1=1}^{4} e^{i\lambda_i t} <3+4|\lambda_i><\lambda_i|1> \\ &= 4\frac{(2^{1/2}J_1J_2)^2}{\lambda_\pm^4}\sin^4(\lambda_\pm t/2). \quad (413)\end{aligned}$$

The factor before $\sin^4(\lambda_\pm t/2)$ is rewritten as

$$4\frac{(2^{1/2}J_1J_2)^2}{\lambda_\pm^4} = 4\frac{2J_1^2J_2^2}{(J_1^2+2J_2^2+2\delta^2)^2}. \quad (414)$$

Since

$$4\frac{2J_1^2J_2^2}{(J_1^2+2J_2^2+2\delta^2)^2} \le 4\frac{2J_1^2J_2^2}{(J_1^2+2J_2^2)^2} \le 1, \quad (415)$$

the maximum efficiency unity is attained when the chain is symmetric ($\delta = 0$) and $J_1 = 2^{1/2}J_2$. If the condition $J_1 = 2^{1/2}J_2$ is kept, the reduction of the efficiency for small asymmetry is given by

$$\frac{1}{(1+\delta^2/2J_2^2)^2} \sim 1 - \delta^2/J_2^2. \quad (416)$$

Thus the reduction is proportional to the second order of the asymmetry $(\delta/J_2)^2$.

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APPENDIX

The Hamiltonian matrix is written as

$$\mathsf{A} = \begin{pmatrix} 0 & J_1 & 0 & & 0 & 0 & 0 \\ J_1 & 0 & J_2 & & 0 & 0 & 0 \\ 0 & J_2 & 0 & & 0 & 0 & 0 \\ . & . & . & . \; . & . & . & . \\ 0 & 0 & 0 & & 0 & J_{N-2} & 0 \\ 0 & 0 & 0 & & J_{N-2} & 0 & J_{N-1} \\ 0 & 0 & 0 & & 0 & J_{N-1} & 0 \end{pmatrix}, \tag{A.1}$$

where

$$J_1 = J_{N-1} = a. \tag{A.2}$$

We denote the inner $(N-2)$ dimensional matrix by A'

$$\mathsf{A}' = \begin{pmatrix} 0 & J_2 & 0 & & 0 & 0 & 0 \\ J_2 & 0 & J_3 & & 0 & 0 & 0 \\ 0 & J_3 & 0 & & 0 & 0 & 0 \\ . & . & . & . \; . & . & . & . \\ 0 & 0 & 0 & & 0 & J_{N-3} & 0 \\ 0 & 0 & 0 & & J_{N-3} & 0 & J_{N-2} \\ 0 & 0 & 0 & & 0 & J_{N-2} & 0 \end{pmatrix} \tag{A.3}$$

and assume that eigenstates $\{|u^{\lambda_i} >\}_{i=1,\ldots,N-2}$ and eigenvalues $\{\lambda_i\}_{i=1,\ldots,N-2}$ of A' are known:

$$\mathsf{A}'|u^{\lambda_i} >= \lambda_i |u^{\lambda_i} > \tag{A.4}$$

$$|\lambda_i >= \begin{pmatrix} u_1^{\lambda_i} \\ u_2^{\lambda_i} \\ \\ \\ u_{N-3}^{\lambda_i} \\ u_{N-2}^{\lambda_i} \end{pmatrix}. \tag{A.5}$$

When $|a| \ll 1$, we have two eigenstates of A, $|\pm >>$, which have eigenvalues close to zero, $\pm\lambda_0$:

$$\mathsf{A}|\pm \gg = \pm\lambda_0 |\pm \gg, \tag{A.6}$$

$$\lambda_0 = a^2 \sum_{i=1}^{N-2} \frac{1}{\lambda_i} u_1^{\lambda_i} u_{N-2}^{\lambda_i} + \mathcal{O}(a^3), \tag{A.7}$$

$$|\pm \gg = \begin{pmatrix} 2^{-1/2} + \mathcal{O}(a^2) \\ \mathcal{O}(a) \\ . \\ . \\ \mathcal{O}(a) \\ \mp 2^{-1/2} + \mathcal{O}(a^2) \end{pmatrix}. \tag{A.8}$$

Other $(N-2)$ eigenvalues and eigenstates are given by

$$\mathsf{A}|i \gg = (\lambda_i + \mathcal{O}(a^2))|i \gg, \tag{A.9}$$

$$|i \gg = \begin{pmatrix} a u_1^{\lambda_i}/\lambda_i + \mathcal{O}(a^2) \\ u_1^{\lambda_i} + \mathcal{O}(a^2) \\ \\ \\ u_{N-2}^{\lambda_i} + \mathcal{O}(a^2) \\ a u_{N-2}^{\lambda_i}/\lambda_i + \mathcal{O}(a^2) \end{pmatrix}, \qquad i = 1, \ldots, N-2. \tag{A.10}$$