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Estimation Method of the Optimum Relaxation Factor for the Successive Overrelaxation Method

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SYNOPSIS

New estimation method of the optimum relaxation factor for the successive overrelaxation method (SOR) is proposed, and the efficiency of the new method is surveyed through a number of numerical experiments. This method can a priori determine the value of the factor by using only the topological properties of the problem, and it is valid for a sparse set of linear equations obtained by using the five-point difference scheme for any rectangular area with arbitrary boundary conditions. The experiments clarify that the method can estimate good approximate value of the factor.

1. INTRODUCTION

Though the iterative methods for a large sparse set of linear equations require the minimum amount of memory neccessary for storing only the nonzero elements, their execution-time neccessary for the computations is generally longer than the direct methods which are based on the elimination method. But, it is also well known that the successive overrelaxation method (SOR) which is one of the iterative methods can oftenly save its execution-time for a large amount comparing to that of the Gauss-Seidel method (GS) which is also another iterative one. For this saving of the computation time so-called the relaxation factor, ω, used in the SOR method must be appropriately determined, and

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a number of methods are already proposed[1, 2].

For linear systems with a certain property this relaxation factor is expressed as function of the spectral radius of the SOR matrix, and, therefore, the problem to determine the optimum value of the factor is replaced by how to search the spectral radius. Since the spectral radius is governed by the maximum eigenvalue of the SOR matrix, the optimum value, ω_{opt}, is exactly determined for a linear system obtained by using the five-point difference scheme for following problem;

Seek a function u(x,y) continuous in R+S, which is twice continuously differentiable in R and which satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = G(x,y) \text{ in R, and} \tag{1}$$

$$u(x,y) = g(x,y) \text{ on S} \tag{2}$$

, where G(x,y) and g(x,y) are contimuous functions defined in R and S, respectively, where R is the interior and S is the boundary of the unit square $0 < x < 1$, $0 < y < 1$.

In this case the spectral radius is strictly obtained as following equation;

$$\frac{1}{2}\left(\cos \frac{\pi}{n} + \cos \frac{\pi}{m} \right) \tag{3}$$

, where n $(=1/h_1)$ and m $(=1/h_2)$ are the number of subdivisions along x and y axes, respectively. (h_1 and h_2 are, therefore, the width of one subdivision along x and y axes, respectively.) But, (3) is valid only for above case, and the determination of the spectral radius is generally very difficult. For example, if the boundary condition, i.e. (2), is replaced by

$$u(x,y) = g(x,y) \text{ on } S_1 \subset S \tag{4}$$

, then (3) cann't estimate the spectral radius for this case strictly.

According to the theoretical study on the convergence rate of the Gauss-Seidel method the convergence rate is directly related to the spectral radius. On the other hand, one of the authors of this paper has already proposed an estimation method of the convergence rate of the GS method for any rectangular area subdivided by the five-point difference sheme. Since the estimation method is valid for problem with any type of boundary conditions, that is, it can be applied , for example, for a rectangular area supported at one edge, the introduction of this estimation method to the calculation of the spectral radius directly leads to the proposal of a new estimation method of ω_{opt} [3].

The main purpose of this paper is the proposal of a new estimation method of the optimum relaxation factor for the SOR method valid for any linear system obtained by the five-point difference scheme applied

for a rectangular area.

In Chapter 2 the relations between fundamental iterative procedures including the GS and SOR methods are explained, and in Ch.3 a new estimation method of the relaxation factor is obtained by using the concepts presented in the preceding chapter. The efficiency of the method is examined through a number of numerical experiments, and the results are presented in Chapter 4.

2. SUCCESSIVE OVERRELAXATION METHOD

Let

$$A\,u = b \tag{5}$$

be a large sparse set of linear equations directly obtained by using the five-point difference scheme for the problem presented in the introduction, that is , (1) and (4). Then, A is possitive-definite and symmetric.

From the i-th equation of (5) we obtain

$$u_i = \Big(b_i - \sum_{j=1}^{i-1} a_{ij}u_j - \sum_{j=i+1}^{N} a_{ij}u_j \Big)/a_{ii} \tag{6}$$

, where a_{ij}, u_i and b_i are the elements of A, u and b in (5), respectively.

By giving an appropriately selected initial value for u, namely $u^{(0)}$, (6) gives a new solution vector, $u^{(1)}$. By repeting this procedure $u^{(k+1)}$ is expressed as following;

$$u_i^{(k+1)} = \Big(b_i - \sum_{j=1}^{i-1} a_{ij}u_j^{(k)} - \sum_{j=i+1}^{N} a_{ij}u_j^{(k)} \Big)/a_{ii} \tag{7}$$

$u_j^{(k)}$ for $j < i$ in (7) may be replaced by $u_j^{(k+1)}$, because $u_j^{(k+1)}$ are already obtained at the i-th stage. Then,

$$u_i^{(k+1)} = \Big(b_i - \sum_{j=1}^{i-1} a_{ij}u_j^{(k+1)} - \sum_{j=i+1}^{N} a_{ij}u_j^{(k)} \Big)/a_{ii} \tag{8}$$

(7) and (8) are the Jacobi and the Gauss-Seidel methods, respectively.

Let D, E and F be the main diagonal, strictly lower and strictly upper triangular matrix of A, respectively. That is,

$$A = D - E - F \tag{9}$$

Then, the J and GS methods are rewritten as followings;

$$u^{(k+1)} = - D^{-1}(E + F) u^{(k)} + D^{-1}b \tag{10}$$

$$u^{(k+1)} = (D - E)^{-1}Fu^{(k)} + (D - E)^{-1}b \tag{11}$$

By using the GS method we obtain $\hat{u}_i^{(k+1)}$ as following;

$$a_{ii}\hat{u}_i^{(k+1)} = b_i - \sum_{j=1}^{i-1} a_{ij}u_j^{(k+1)} - \sum_{j=i+1}^{N} a_{ij}u_j^{(k)} \tag{12}$$

Then, the (k+1)st approximate solution, $u_i^{(k+1)}$, is defined as

$$\begin{aligned} u_i^{(k+1)} &= u_i^{(k)} + \omega \{ \hat{u}_i^{(k+1)} - u_i^{(k)} \} \\ &= (1 - \omega)u_i^{(k)} + \omega\hat{u}_i^{(k+1)} \end{aligned} \tag{13}$$

, where ω is a relaxation factor by proper determination of which the rate of convergence is acceralated.

The substitution of (13) to (12) yields to

$$\begin{aligned} a_{ii}u_i^{(k+1)} = a_{ii}u_i^{(k)} + \omega\{ &- \sum_{j=1}^{i-1} a_{ij}u_j^{(k+1)} - \sum_{j=i+1}^{N} a_{ij}u_j^{(k)} \\ &+ b_i - a_{ii}u_i^{(k)} \} \end{aligned} \tag{14}$$

The matrix expression of (14) is given as following;

$$\begin{aligned} u^{(k+1)} = &(I - \omega L)^{-1}\{ (1 - \omega)I + \omega U \}u^{(k)} \\ &+ \omega(I - \omega L)^{-1}D^{-1}b \end{aligned} \tag{15}$$

, where $L = D^{-1}E$ and $U = D^{-1}F$. (14) and (15) are the general form of the point successive overrelaxation method (SOR). Note that if we set $\omega = 1$ in (15), then (15) coincides with the expression of the GS method, i.e. (11).

Following matrices in (7), (8) and (15),

$$B = D^{-1}(E + F) \tag{16}$$

$$C = (D - E)^{-1}F \tag{17}$$

$$P = (I - \omega L)^{-1}\{ (1 - \omega)I + \omega U \} \tag{18}$$

are called as the point Jacobi, point Gauss-Seidel and point successive overrelaxation matrix, respectively.

It is obvious that all of these iterative methods are expressed as following;

$$u^{(k+1)} = Mu^{(k)} + g \tag{19}$$

, where M indicates B, C or P, and g is the second term of above three equations, i.e. (10), (11) or (15). Then,

$$u^{(k+1)} = M^k u^{(1)} + (M^{k-1} + M^{k-2} + \ldots + M^2 + M + I)g \tag{20}$$

, where I is a unit matrix.

Here, we introduce the definition of a matrix norm expressed as following;

$$|| M || = \sup_{x \neq 0} \frac{|| Mx ||}{|| x ||} \tag{21}$$

, where $|| x ||$ is a vector norm equal to the length of the vector x, i.e. $(\Sigma |x_i|^2)^{1/2}$. In above expression $|| M ||$ is the spectral norm of the matrix, M.

By introducing (21) into (20) we obtain that if $|| M || < 1$, then for $k \to \infty$ the solution vector $u^{(k)}$ tends to a convergent vector presented by

$$u = (I - M)^{-1} g \tag{22}$$

That is, if $|| B ||$, $|| C ||$ or $|| P || < 1$, then above three iterative methods converge to a strict solution vector, and the value of the matrix norm decides the number of iterations neccessary to obtain the convergent solution with sufficient accuracy.

Let's assume to solve the model problem in Ch.1 by using above three iterative methods. Actually the number of iterations of the Jacobi method required to achieve a specific degree of convergence is twice time of that of the GS method[2], and the number by the SOR method for optimally selected relaxation factor oftenly decreases till few percent of the number of iterations of the GS method. These differences depend on the value of the matrix norms of three convergent matrices, i.e. (16), (17) and (18). This suggests that as a solver of a large sparse set of linear equations the SOR method is sufficiently effective, rational and economical comparing to the direct method like the Band Solver or the Skyline Solver, because the SOR method requires only nonzero elements of the original matrix, A, and the memory size is not influenced by the elimination ordering and also by the configuration of the original system. Moreover, it is very tough for the calculation error which gives important effect to direct methods.

3. DETERMINATION OF OPTIMUM RELAXATION FACTOR

In preceding chapter it is shown that the characteristics of the convergence of any iterative method is governed by the spectral norm of the convergence matrix. Especially, in case of the SOR method the norm depends on the value of the relaxation factor, and in order to acceralate the convergence ratio the norm must be minimized by appropriate selection of the factor. The main purpose of this chapter is the proposal of the estimation method of the factor for any sparse set of linear equations which is obtained by use of the five-point difference scheme for any rectangular area.

Let M be one of the convergence matrix, i.e. B, C and P. Then, the spectral radius of a matrix, M, is defined as

$$S(M) = \max_{\lambda \in S_M} |\lambda| \tag{23}$$

, where S_M is the set of all eigenvalues of M[2].

Since the convergence matrices are positive-definite in our objective problem, then S(M) is equal to the maximum eigenvalue of M. That is, (23) is replaced to following equation;

$$S(M) = \max_{\lambda \in S_M} |\lambda| = \lambda_{max} \tag{24}$$

According to [2], the value of the relaxation factor which is optimum in the sense of minimizing P is given by

$$\omega = \frac{2}{1 + (1 - S(B)^2)^{1/2}} \tag{25}$$

, where S(B) is the spectral norm of the Jacobi method.

By considering

$$S(C) = S(B)^2 \tag{26}$$

, then (25) is rewritten as following;

$$\omega = \frac{2}{1 + (1 - S(C))^{1/2}} \tag{27}$$

(25) and (27) suggest that if the spectral radii of the point Jacobi and the point GS matrices are obtained, the optimum value of ω for P is calculated.

The case of the model problem presented in Ch. 1 is a good example where ω_{opt} can be a priori estimated by using (25). But, the estimation method of the spectral radius of B, i.e.(3), is valid only for the case of the boundary condition (2), and, therefore, for the other cases where only a part of u on the boundary is given as (4) and the others are not prescribed, the estimation of S(B) by using (3) is not valid.

Assume the bending problem of a rectangular plate-like structure. If the plate is fixed at four edges which surround the plate, the estimation of λ_{max}, and, therefore, that of ω_{opt}, is strictly obtained by using (3) and (25). But, for the other boundary conditions, above estimation cann't be directly applied.

Let $\varepsilon^{(n)}$ and $\varepsilon^{(m)}$ be error vectors after n and m iterations of the GS method applied to (5). That is,

$$\begin{aligned} \varepsilon^{(n)} &= u^{(n)} - u \\ \varepsilon^{(m)} &= u^{(m)} - u \end{aligned} \tag{28}$$

, where u is a strict solution vector. For n > m, we assume following relation for these two error vectors;

$$\frac{||\varepsilon^{(n)}||}{||\varepsilon^{(m)}||} = \frac{1}{10} \tag{29}$$

From (20),

$$\begin{aligned} ||\varepsilon^{(n)}|| = ||\, C\varepsilon^{(n-1)}\,|| &\leqslant ||C||\,||\varepsilon^{(n-1)}|| \\ &\leqslant ||C||^2\,||\varepsilon^{(n-2)}|| \\ &\cdots\cdots\cdots \\ &\leqslant ||C||^{n-m}\,||\varepsilon^{(m)}|| \end{aligned} \tag{30}$$

Substitution of (30) into (29) leads to following expression.

$$S(C)^{n-m} = ||C||^{n-m} \geqslant 10^{-1} \tag{31}$$

Since C is a convergence matrix, the spectral radius, S(C), is less than 1. Then, from (31) we can obtain δIT (= n - m) which satisfies following equation;

$$S(C) = o.1^{1/\delta IT} \tag{32}$$

Then, the substitution of (32) into (27) leads to following new expression of ω;

$$\omega = \frac{2}{1 + (1 - 0.1^{1/\delta IT})^{1/2}} \qquad (33)$$

(33) indicates that if δIT which is the number of iterations neccessary for raising the accuracy by one figure is given, then ω is easily estimated.

On the other hand, Taniguchi and Kanei proposed following equation for the estimation of δIT which is valid for the problem in Chapter 1[3];

$$\delta IT = \frac{k(2k^2 + k + 1)}{4(k+1)(k^2 + 1)} B^2 \qquad (34)$$

, where k and B are the ratio of two edges (H/B) and the width of the rectangular area, respectively. (See Fig.1) Then, the substitution of (34) into (33) gives the final form to estimate ω_{opt}.

$$\omega = \frac{2}{1+(1-0.1^{\frac{4(k+1)(k^2+1)}{k(2k^2+k+1)B^2}})^{1/2}} \qquad . \ . \ . \qquad (35)$$

(35) indicates that ω_{opt} is estimated when the topological properties of the problem area are presented. That is, (35) is a new a-priori estimation method of ω_{opt}.

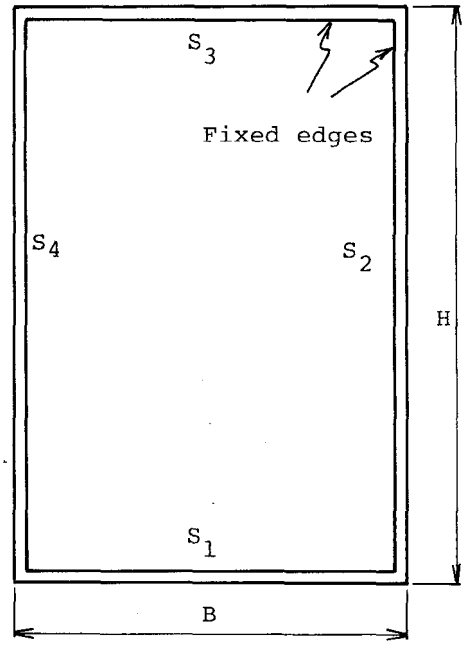

Fig.1 Model Problem

The boundary conditions for a rectangular area are fundamentally classified into following five types;

Type 1 : All u for S_1, S_2, S_3 and S_4 are prescribed.
Type 2 : u for three edges are prescribed.
Type 3 : u for adjacent two edges are prescribed.
Type 4 : u for opposite two edges are prescribed.
Type 5 : u for only one edge are prescribed.

The estimation method of δIT by [3] is to estimate the number of iterations, δIT, for a rectangular area with Type 1 boundary condition (H*B) which is obtained from original rectangular area (H'*B') of the other boundary condition type by following procedure.

1). Type 1 Boundary Condition

Set H = H' and B = B', and apply (35).

2). Type 2 Boundary Condition

If H' < B', then set H = 2*H' and B = B', and apply (35). (See Fig.2-a)

If H' ≥ B', then set H = H'+B' and B = B', and apply (35). (See Fig.2-b)

3). Type 3 Boundary Condition

Set H = 2*H' and B = 2*B', and apply (35). (See Fig.3)

4). Type 4 Boundary Condition

If H' > 4*B', then set H = H' and B = B', and apply (35). (See Fig.4-a)

If H' ≤ 4*B', then set H = 4*B' and B = B', and apply (35). (See Fig.4-b)

5). Type 5 Boundary Condition

Set B = 2*B'. If H' > 4*B, then set H = H' and B = 2*B', and apply (35). (See Fig.5-a)

If H' ≤ 4*B, then set H = 4*B and B = 2*B', and apply (35). (See Fig.5-b)

Note that in above procedures "B" is the length of the shorter edge, and "H" is the longer one of the transformed rectangular area.

In order to estimate δIT B and H may be introduced in (34).

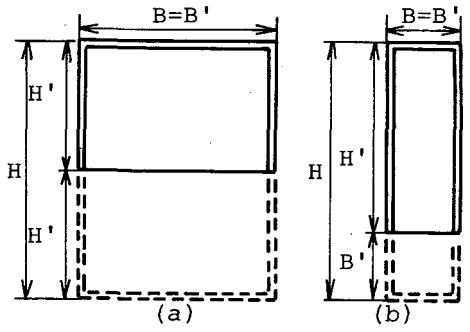

Fig.2 Treatment of Type 2 B.C.

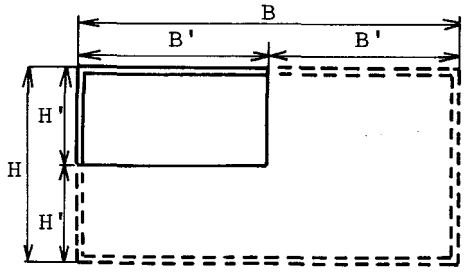

Fig.3 Treatment of Type 3 B.C.

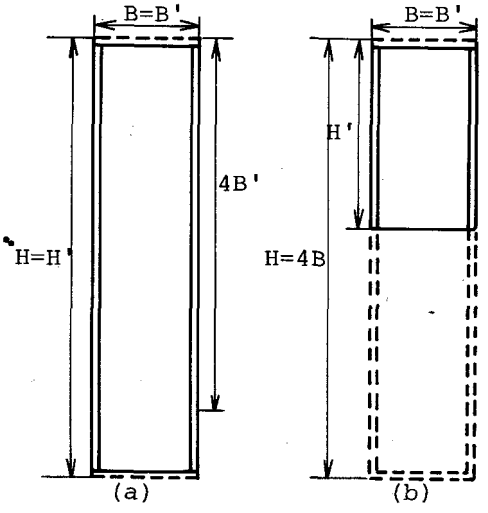

Fig.4 Treatment of Type 4 B.C.

4. NUMERICAL EXPERIMENTS

Since the estimation of δIT by (34) includes less than 10%error, the influence of this error to the value of the estimated optimum relaxation factor must be examined. This numerical experiments are done by the comparison of ω-values by substituting δIT and 1.1*δIT into (34).

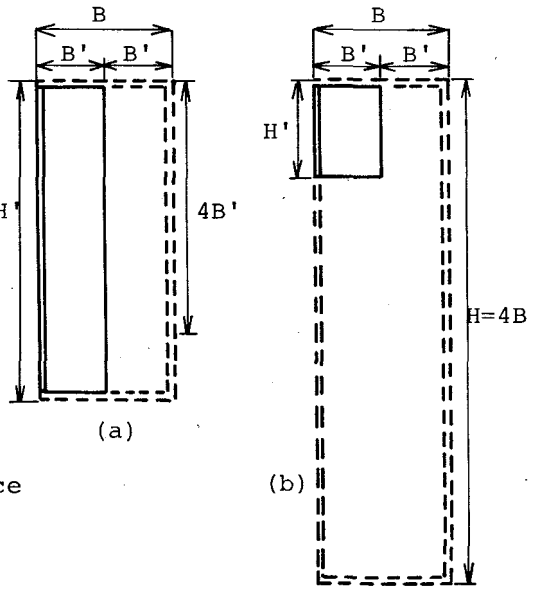

Fig.5 Treatment of Type 5 B.C.

As obvious from the results summarized in Table 1, the difference between $\omega_{\delta IT}$ and $\omega_{1.1\delta IT}$ is large for small value of δIT. But, for the case where δIT is small, total amount of iterations for the approximate solution is small, and, therefore, the difference does not give important effect to the actual iterative computation. On the other hand, we recognize that for cases of rather large δIT the difference between $\omega_{\delta IT}$ and $\omega_{1.1\delta IT}$ becomes small.

In order to survey the influence of this difference to the number of iterations a number of test examples must be actually solved by using the SOR method. Test examples treated in this chapter are constructed by following procedure; We consider two-dimensional area R surrounded by S which are rectangular $0 < x < a$, $0 < y < b$ where for some $h > 0$ the quantities a/h and b/h are integers, namely NG1-1 and NG2-1, respectively.

For any value of h such that a/h and b/h are integers, we have

Case	δIT	λ_{max}	ω_{opt}
1	50	0.9550	1.6550
	55	0.9590	1.6632
2	100	0.9772	1.7378
	110	0.9793	1.7484
3	200	0.9886	1.8067
	220	0.9896	1.8148
4	500	0.9954	1.8730
	550	0.9958	1.8786
5	1000	0.9977	1.9085
	1100	0.9979	1.9125
6	2000	0.9988	1.9344
	2200	0.9990	1.9373
7	5000	0.9995	1.9580
	5500	0.9996	1.9599
8	10000	0.9998	1.9701
	11000	0.9998	1.9715

Table 1 Influence of δIT to ω

Note ; Upper row for δIT
Lower row for 1.1δIT

the difference equation

$$4u(x,y) - u(x+h,y) - u(x,y+h) - u(x-h,y) - u(x,y-h) = 0 \tag{36}$$

for any interior nodal point, (x,y). For a node on S which is connected to the boundary the difference equation is obtained by equating one or two terms between the second and fifth in (36) to zero. For nodes on S which are not connected to the boundary the difference equation is obtained by replacing the coefficient of the first term by "3" or "2" and deleting one or two terms between the second and the fifth in (36) for the edge and the corner node, respectively. That is, for an edge node we have, for example,

$$3u(x,y) - u(x+h,y) - u(x,y+h) - u(x-h,y) = 0 \tag{37}$$

, and for a corner node we obtain, for example,

$$2u(x,y) - u(x+h,y) - u(x,y+h) = 0 \tag{38}$$

Unit load is applied for all nodes in the area, R ∪ S. By this procedure we obtain a set of linear equations

$$A\,u = b \tag{39}$$

where A is a (NG1*NG2) * (NG1*NG2) coefficient matrix, and b is a unit vector.

As the boundary conditions five types which are explained in preceding chapter are considered, and for each type several test examples are appropriately selected and used for the experiments. A part of the results are presented in Appendix.

From these results we may conclude that the new estimation method of the relaxation factor can guess rather good value of ω, even though the estimation of δIT includes about less than 10% of error, and, therefore, this method is valid for engineering problems.

5. CONCLUDING REMARKS

In this paper a priori determination of the relaxation factor for the successive overrelaxation method is proposed, and the accuracy of the estimated values is examined through a number of numerical experiments. The results show that the estimated values of ω_{opt} are near the true optimum even though it includes some error, and we may

conclude that the new method is sufficiently valid for engineering problems. Especially, since the proposed method can treat any problem with various types of the boundary conditions and it is a method of the a priori determination of the relaxation factor, it is more valid comparing to other methods presently in use.

In order to improve the accuracy of this estimation method we may improve the estimation of δIT, and for this purpose more numerical experiments for the estimation of δIT must be continued.

At present, new method is valid only for a set of linear equations obtained from the five-point difference scheme applied for a rectangular region, and, therefore, its extension for any finite element mesh system for a rectangular region and, furthermore, for any region without constraints of the boundary configuration is required.

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APPENDIX : RESULTS OF NUMERICAL EXPERIMENTS

In tables following symbols are used;

NG1 ; Number of nodes on vertical edge

NG2 ; Number of nodes on lateral edge

δIT ; Number of iterations estimated by (34)

dIT ; Actual number of iterations by the experiment

IT ; Total number of iterations by the experiment

IT0 ; Total number of iterations by the GS method

IT1 ; Total number of iterations by the SOR method for estimated ω_{opt}

IT2 ; Total number of iterations by the SOR method for true ω_{opt}

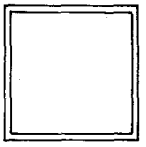
ω_{opt} ; Relaxation factor estimated by (35)

H' ; Length of the vertical edge of the rectangular area
B' ; Length of the lateral edge of the rectangular area
H ; Length of the longer edge obtained by the procedure in Ch.3
B ; Length of the shorter edge obtained by the procedure in Ch.3
k ; Value of H/B

Table 2

Number of Iterations for Type 1 Boundary Condition

NG1=40 H'=41 H=41 k=41/31
NG2=30 B'=31 B=31
$\delta IT=f(k)B^2=290$ $\omega_{opt}=1.835$



ω	dIT							IT
	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	
1.00	260	286	285	284	285	285	285	1970
1.81	55	23	23	22	23	23	23	192
1.82	52	20	20	20	20	21	20	173
1.83	50	17	16	16	17	17	17	150
1.84	49	19	5	9	18	11	17	128
1.85	47	25	2	15	16	10	19	134
1.86	53	20	2	18	14	19	18	144
1.87	58	15	9	20	15	21	8	146

IT1/IT0=IT2/IT0=0.065

Table 3

Number of Iterations for Type 2 Boundary Condition

NG1=10 H'=10 B=20 k=99/20
NG2=100 B'=99 H=99
$\delta IT=f(k)B^2=180$ $\omega_{opt}=1.797$

ω	dIT							IT
	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	
1.00	160	195	192	189	189	188	188	1301
1.77	41	24	22	22	20	20	19	168
1.78	40	21	22	20	19	17	18	157
1.79	38	20	21	19	16	15	15	144
1.80	36	19	19	18	16	10	8	126
1.81	34	18	17	18	17	19	7	130
1.82	32	20	20	20	20	15	8	135
1.83	35	22	19	20	20	14	14	144

IT1/IT0=IT2/IT0=0.097

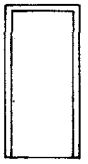
Table 4

Number of Iterations for Typ2 2 Boundary Condition

NG1=100 H'=100 H=111 k=111/11

NG2=10 B'=11 B=11

$\delta IT=f(k)B^2=57$ $\omega_{opt}=1.668$



ω	dIT							IT
	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	
1.00	47	56	57	56	57	56	57	386
1.64	20	20	11	11	11	11	11	85
1.65	19	11	10	10	11	10	11	82
1.66	19	10	10	10	10	10	10	79
1.67	19	9	10	9	10	9	10	76
1.68	18	9	10	9	9	10	9	74
1.69	18	9	9	9	10	11	10	76
1.70	18	9	9	10	11	11	11	79

IT1/IT0=0.200 IT2/IT0=0.192

Table 5

Number of Iterations for Type 2 Boundary Condition

NG1=30 H'=30 H=60 k=60/41

NG2=40 B'=41 B=41

$\delta IT=536$ $\omega_{opt}=1.877$

ω	dIT							IT
	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	
1.00	480	532	531	531	531	531	532	3668
1.85	78	34	34	34	34	34	34	282
1.86	72	29	30	30	30	29	30	250
1.87	66	24	25	24	24	25	24	212
1.88	62	16	14	16	25	18	19	170
1.89	70	8	19	18	27	17	18	177
1.90	72	13	20	26	22	20	20	193
1.91	72	22	21	28	28	17	25	213

IT1/IT0=IT2/IT0=0.046

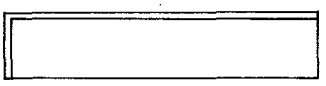
Table 6

Number of Iterations for Type 3 Boundary Condition

NGl=10 H'=10 H=200 k=200/20=10

NG2=100 B'=100 B=20

δIT=190 ω_{opt}=1.802



ω	dIT							IT
	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	
1.00	161	195	195	194	194	193	193	1325
1.77	42	23	23	22	22	21	20	173
1.78	41	21	22	20	21	19	18	162
1.79	40	19	20	20	18	17	16	150
1.80	38	19	19	18	17	13	12	136
1.81	37	18	18	18	17	13	12	133
1.82	36	18	17	18	18	20	11	138
1.83	35	18	21	19	21	17	12	143

IT1/IT0=0.103 IT2/IT0=0.100

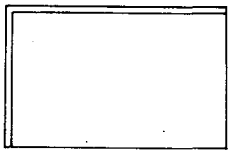
Table 7

Number of Iterations for Type 3 Boundary Condition

NGl=30 H'=30 H=80 k=80/60

NG2=40 B'=40 B=60

δIT=1090 ω_{opt}=1.912



ω	dIT							IT
	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	
1.00	974	1076	1076	1076	1076	1076	1076	7430
1.88	126	58	58	58	58	58	58	474
1.89	114	50	51	50	51	50	50	416
1.90	101	42	42	42	42	42	42	353
1.91	88	30	30	31	30	30	31	270
1.92	76	34	27	26	30	27	21	241
1.93	88	27	39	25	36	30	33	278
1.94	99	38	39	33	41	33	33	316

IT1/IT0=0.036 IT2/IT0=0.032

Table 8

Number of Iterations for Type 4 Boundary Condition

NG1=10 H'=9 H=244 k=244/61=4

NG2=60 B'=61 B=61

$\delta IT=1619$ $\omega_{opt}=1.927$

ω	dIT							IT
	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	
1.00	1366	1649	1649	1649	1650	1649	1649	11261
1.90	172	74	74	74	74	74	74	616
1.91	154	62	63	63	62	63	62	529
1.92	133	50	50	49	50	49	50	431
1.93	110	28	24	17	53	37	29	298
1.94	107	36	45	30	44	32	41	335
1.95	118	46	58	35	39	51	48	395
1.96	154	50	43	76	64	53	50	490

IT1/IT0=IT2/IT0=0.026

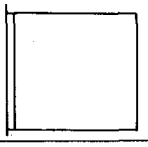
Table 9

Number of Iterations for Type 5 Boundary Condition

NG1=50 H'=19 H=400 k=4

NG2=50 B'=50 B=100

$\delta IT=4353$ $\omega_{opt}=1.955$



ω	dIT							IT
	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	
1.00	3755	4594	4593	4594	4594	4593	4594	31317
1.940	298	120	120	121	120	120	120	1019
1.950	239	88	87	88	87	88	88	765
1.951	232	84	84	84	84	83	84	735
1.952	225	80	80	80	80	80	79	704
1.953	218	76	75	76	75	76	76	672
1.954	211	71	70	71	71	71	71	636
1.955	203	65	65	66	65	65	65	594
1.956	194	59	58	58	57	57	57	540
1.957	186	52	46	40	37	72	63	496
1.958	176	43	31	90	55	46	41	482
1.959	169	34	88	50	42	62	65	510
1.960	163	73	56	44	65	60	43	504
1.970	226	51	95	73	83	45	83	656
1.980	308	127	103	117	114	97	140	1006
1.990	636	216	214	225	225	214	269	1999

IT1/IT0=0.019 IT2/IT0=0.015