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Galerkin Method to an Integral Equation in the Kinetic Theory

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SYNOPSIS

A Galerkin method is applied to a singular integral equation of Fredholm type originated in the problem of the rarefied gas flow over a plane wall. The solution is expanded in a series of the Abramowitz function. The numerical calculations were carried out up to ten-terms expansions. The results show a good convergence of the series. The comparison is made with the previous work obtained by the moment method.

1. INTRODUCTION

A lot of work has been done for the flow of rarefied gas, e.g., the shear flow,[1,2] Couette flow,[2,3] the plane Poiseuille flow,[4] the cylindrical Poiseuille flow,[5,6] the flow past a circular cylinder,[7,8] the heat transfer from a circular cylinder,[9,10] the flow past a sphere,[11,12] the plane shock wave,[13] etc.. These studies are carried out mainly on the basis of the Bhatnager - Gross - Krook model (BGK model)[14] of the Boltzmann equation. The BGK model equation, which is a nonlinear integro - differential equation for the distribution function of the molecular velocity, is reduced to the simultaneous singular integral equations of Fredholm type for

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the density, flow velocity and temperature. An exact solution has not been obtained even for a simple case of the equation. We must appeal to a numerical calculation to get a solution.
One of the powerful method for a numerical solution may be the moment method.[15, 16, 17] Another useful method will be the collocation method. This is simpler than the moment method and therefore it is applicable to more complicated equations.[10]

In the present paper, we shall apply the Galerkin method to the singular integral equation appeared in the problem of the high speed flow of rarefied gas over a plane wall.[2] The solution is expanded in a series of the Abramowitz function. The numerical solution is compared with the result obtained by the moment method.

2. FUNDAMENTAL EQUATION

The integral equations to be solved are given by

$$\pi^{1/2}\Delta_1 = L_0(\Delta_1) - \varkappa_1 J_0 + J_1 , \tag{1}$$

$$\pi^{1/2}\Omega_1 = L_1(\Omega_1, \Theta_1) - \alpha_1(J_2 - J_0) + J_3 - \frac{3}{2} J_1 , \tag{2a}$$

$$\frac{3}{2}\pi^{1/2}\Theta_1 = L_2(\Omega_1, \Theta_1) - \alpha_1(J_4 - \frac{3}{2} J_2 + \frac{3}{2} J_0) + J_5 - 2J_3 + \frac{7}{4} J_1 \tag{2b}$$

$$\pi^{1/2}\Delta_2^{(2)} = L_0(\Delta_2^{(2)}) + Ih_0^{(2)} , \tag{3}$$

$$\begin{aligned} Ih_0^{(2)} = & - \varkappa_2^{(2)} J_0 - \pi^{1/2}\{(x + \varkappa_1 + \Delta_1)\Omega_1 - x\Delta_1\} - 2J_4 + \varkappa_1 J_3 \\ & + 3J_2 - \frac{3}{2}\varkappa_1 J_1 - \frac{1}{2}\pi^{1/2}\alpha_1 \Delta_1 + \int_0^\infty \{(y + \varkappa_1 + \Delta_1)\cdot \\ & [\Omega_1 J_{-1}(|x - y|) + \Theta_1 \{J_1(|x - y|) - \frac{1}{2} J_{-1}(|x - y|)\}] \\ & + y\Delta_1 [J_1(|x - y|) - \frac{3}{2} J_{-1}(|x - y|)]\}dy , \end{aligned} \tag{4}$$

$$\pi^{1/2}\Omega_2^{(i)} = L_1(\Omega_2^{(i)}, \Theta_2^{(i)}) + Ih_1^{(i)} , \tag{5a}$$

$$\frac{3}{2}\pi^{1/2}\Theta_2^{(i)} = L_2(\Omega_2^{(i)}, \Theta_2^{(i)}) + Ih_2^{(i)} ; \quad (i=2,3) \tag{5b}$$

$$Ih_1^{(2)} = - \alpha_2^{(2)}(J_2 - J_0) + \frac{2}{5}(2J_4 - 3J_2 - J_0) , \tag{6a}$$

$$Ih_2^{(2)} = - \alpha_2^{(2)}(J_4 - \frac{3}{2} J_2 + \frac{3}{2} J_0) + \frac{4}{5} J_6 - \frac{8}{5} J_4 - J_2$$

$$+ \frac{1}{5} (1 + 5\varkappa_1^2)J_0 - \pi^{1/2}(2x + \Delta_1)\Delta_1 + L_0\{(2y + \Delta_1)\Delta_1\} \quad , \tag{6b}$$

$$Ih_1^{(3)} = - \alpha_2^{(3)}(J_2 - J_0) - J_6 + \alpha_1 J_5 + \frac{1}{2} (10 - \alpha_1^2)J_4 - 4\alpha_1 J_3$$

$$+ \frac{1}{4} (6\alpha_1^2 - 15)J_2 + \frac{9}{4} \alpha_1 J_1 - \frac{1}{4} (1 + 2\alpha_1^2)J_0$$

$$+ 2J_0\int_0^\infty \{(y + \alpha_1 + \Theta_1)[\Omega_1 (J_2 - \frac{1}{2} J_0)$$

$$+ \Theta_1 (J_4 - 4J_2 + \frac{5}{4} J_0)] - \frac{1}{2} \Theta_1^2 (J_4 - 5J_2 + \frac{7}{4} J_0)\}dy$$

$$+ \int_0^\infty \{(y + \alpha_1 + \Theta_1)[\Omega_1 \{J_1(|x - y|) - \frac{1}{2} J_{-1}(|x - y|)\}$$

$$+ \Theta_1 \{J_3(|x - y|) - 4J_1(|x - y|) + \frac{5}{4} J_{-1}(|x - y|)\}]$$

$$- \frac{1}{2} \Theta_1^2 [J_3(|x - y|) - 5J_1(|x - y|) + \frac{7}{4} J_{-1}(|x - y|)]\}dy \quad , \tag{7a}$$

$$Ih_2^{(3)} = - \alpha_2^{(3)}(J_4 - \frac{3}{2} J_2 + \frac{3}{2} J_0) + \frac{3}{2} \pi^{1/2}\{(x + \alpha_1)\Theta_1$$

$$- (x + \alpha_1 + \Theta_1)\Omega_1\} - J_8 + \alpha_1 J_7 + \frac{1}{2} (11 - \alpha_1^2)J_6 - \frac{9}{2} \alpha_1 J_5$$

$$+ \frac{1}{4} (7\alpha_1^2 - 33)J_4 + \frac{25}{4} \alpha_1 J_3 + \frac{1}{8} (37 - 18\alpha_1^2)J_2 - \frac{25}{8} \alpha_1 J_1$$

$$+ \frac{1}{8} (1 + 6\alpha_1^2) J_0 + (2J_2 - J_0) \int_0^\infty \{(y + \alpha_1 + \Theta_1)\cdot$$

$$[\Omega_1 (J_2 - \frac{1}{2} J_0) + \Theta_1 (J_4 - 4J_2 + \frac{5}{4} J_0)]$$

$$- \frac{1}{2} \Theta_1^2 (J_4 - 5J_2 + \frac{7}{4} J_0)\}dy + \int_0^\infty (y + \alpha_1 + \Theta_1)\cdot$$

$$[\Omega_1 \{J_3(|x - y|) - J_1(|x - y|) + \frac{5}{4} J_{-1}(|x - y|)\}$$

$$+ \Theta_1 \{J_5(|x - y|) - \frac{9}{2} J_3(|x - y|) + \frac{21}{4} J_1(|x - y|)$$

$$- \frac{21}{8} J_{-1}(|x - y|)\}] - \frac{1}{2} \Theta_1^2 [J_5(|x - y|) - \frac{11}{2} J_3(|x - y|)$$

$$+ \frac{25}{4} J_1(|x - y|) - \frac{31}{8} J_{-1}(|x - y|)]\}dy \quad , \tag{7b}$$

$$L_0(\Delta) = \int_0^{\infty} J_{-1}(|x - y|)\,\Delta dy \quad , \tag{8}$$

$$L_1(\Omega, \Theta) = 2J_0 \int_0^{\infty}\{\Omega J_0 + \Theta\,(J_2 - \tfrac{1}{2} J_0)\}dy$$
$$+ \int_0^{\infty}\{\Omega J_{-1}(|x - y|) + \Theta\,[J_1(|x - y|) - \tfrac{1}{2} J_{-1}(|x - y|)]\}dy \quad , \tag{9}$$

$$L_2(\Omega, \Theta) = (2J_2 - J_0)\int_0^{\infty}\{\Omega J_0 + \Theta(J_2 - \tfrac{1}{2} J_0)\}dy$$
$$+ \int_0^{\infty}\{\Omega\,[J_1(|x - y|) - \tfrac{1}{2} J_{-1}(|x - y|)]$$
$$+ \Theta[J_3(|x - y|) - J_1(|x - y|) + \tfrac{5}{4} J_{-1}(|x - y|)]\}dy \quad , \tag{10}$$

$$J_n(x) = \int_0^{\infty} t^n \exp(- t^2 - \tfrac{x}{t})dt \quad , \tag{11}$$

where the argument of the functions under integral sign is y and that of the other functions is x if it is not shown explicitly. The unknown functions Δ_1, Ω_1, Θ_1, $\Delta_2^{(2)}$, $\Omega_2^{(i)}$, and $\Theta_2^{(i)}$ should vanish faster than x^{-n} for any number n as $x \to \infty$, and $\varkappa_1$, α_1, $\varkappa_2^{(2)}$ and $\alpha_2^{(i)}$ are unknown constants to be determined simultaneously with the solutions. The details of derivation of the above equations are given in ref.(2).

We shall employ a Galerkin method to obtain approximate solutions of eqs.(1), (2), (3) and (5). In this method we assume an expansion of an unknown function, say $\Delta_1(x)$ in eq.(1), in the form :

$$\Delta_1(x) \sim u_N(x) = \sum_{i=0}^{N-1} A_i \psi_i(x) \quad , \tag{12}$$

where the expansion function $\psi_i(x)$ should satisfy the boundary condition at infinity, that is, $\psi_i(x)$ goes to zero faster than x^{-n} for any integer n as $x \to \infty$. The constant N is an arbitrary but fixed positive integer, and A_i is as yet an unknown constant. Let us write eq.(1) symbolically in the form :

$$M(\Delta_1) - f(\varkappa_1) = 0 \quad , \tag{13}$$

where M is the integral operator to the unknown function, and $f(\varkappa_1)$ represents the terms involving $\varkappa_1$ and $J_i(x)$. Then, the constant A_i as well as the unknown constant $\varkappa_1$ is to be determined by the following conditions :

$$\int_0^\infty [M(u_N) - f]\cdot 1dx = 0 \quad , \tag{14a}$$

$$\int_0^\infty [M(u_N) - f]\cdot\psi_k dx = 0 \qquad (k = 0, 1, \ldots, N - 1) \quad . \tag{14b}$$

This gives N + 1 equations for N + 1 unknowns. Since the operator M is linear, the equations obtained are seen to be linear simultaneous equations for A_i and $\varkappa_1$. By solving this, we get an approximate solution of eq.(1). The same discussion can be applied to eq.(3).

The simultaneous integral equations (2a) and (2b) [or (5a) and (5b)] have two unknown functions and one unknown constant. Therefore we need two series expansions, which may have 2N unknown coefficients of the expansion function $\psi_i(x)$. The Galerkin method can be applied to eqs.(2a) and (2b), [or (5a) and (5b)], as before and seems to produce 2N + 2 equations. However, the corresponding equation of eq.(2a) [or (5a)] to eq.(14a) is satisfied identically. Consequently, we obtain 2N + 1 linear simultaneous equations for 2N + 1 unknown constants.

We take the Abramowitz function $J_i(x)$ as an expansion function, which satisfies the condition at large x. That is, we assume the following expansion :

$$u = \sum_{i=0}^{N-1} A_i J_i(x) \quad , \tag{15}$$

where u represents the unknown function Δ_1, $\Delta_2^{(2)}$, Ω_1, Θ_1, $\Omega_2^{(k)}$, or $\Theta_2^{(k)}$, and A_i is the corresponding coefficient.

3. NUMERICAL CALCULATION

The numerical calculations were carried out for N up to 10. All numerical values involved in the linear simultaneous equations to be solved were accurate to about 15 significant digit. Table I shows the variation of Δ_1 versus x for N = 8, 9 and 10. The variations of Ω_1 and Θ_1 are shown in Table II. The values of $\varkappa_1$ and α_1 are listed in Table III. It will be seen that the convergence of the series expansion is quite good. Figures 1 and 2 show the distributions of Δ_1 and (Ω_1, Θ_1),respectively, in which N is taken to be 10.

The integral equations (3) and (5) were solved using the solutions Δ_1, Ω_1, Θ_1, $\varkappa_1$ and α_1 for N = 10. Tables I, III ∿ V show the results obtained. It will be seen that the convergence of the series expansion is satisfactory. The results when N = 10 are plotted in Figs.1,3 and 4.

It may be worth while comparing the present results with the previous work. The solutions of eqs.(1) ∿ (5) based on the moment method, in which the similar expansion of the solution to eq.(15) with N = 4 is used, are given in ref.(2). Some of the results by the moment method are not so good compared with those of the present calculation. These may be replaced by the present solutions. The solutions by the refined moment method are given in refs.(16,17). The present results agree well with the refined solutions.

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Table I. Values of Δ_1 and $\Delta_2^{(2)}$ versus x.

N / x	8	9 Δ_1	10	8	9 $\Delta_2^{(2)}$	10
0.	-0.30906	-0.30907	-0.30907	0.85025	0.85036	0.85036
0.01	-0.29288	-0.29288	-0.29288	0.84284	0.84285	0.84285
0.02	-0.28185	-0.28185	-0.28185	0.83601	0.83602	0.83602
0.03	-0.27263	-0.27263	-0.27263	0.82954	0.82955	0.82955
0.04	-0.26454	-0.26454	-0.26454	0.82334	0.82336	0.82336
0.05	-0.25726	-0.25726	-0.25726	0.81737	0.81739	0.81739
0.10	-0.22827	-0.22827	-0.22827	0.78989	0.78993	0.78993
0.15	-0.20644	-0.20645	-0.20645	0.76523	0.76526	0.76526
0.20	-0.18878	-0.18878	-0.18878	0.74251	0.74254	0.74254
0.25	-0.17392	-0.17392	-0.17392	0.72130	0.72132	0.72132
0.30	-0.16114	-0.16114	-0.16114	0.70132	0.70134	0.70134
0.35	-0.14995	-0.14995	-0.14995	0.68238	0.68240	0.68240
0.40	-0.14003	-0.14003	-0.14003	0.66434	0.66436	0.66436
0.45	-0.13116	-0.13116	-0.13116	0.64710	0.64712	0.64712
0.50	-0.12316	-0.12316	-0.12316	0.63058	0.63060	0.63060
0.60	-0.10929	-0.10929	-0.10929	0.59944	0.59946	0.59946
0.70	-0.09764	-0.09765	-0.09765	0.57050	0.57053	0.57053
0.80	-0.08773	-0.08773	-0.08773	0.54348	0.54351	0.54351
0.90	-0.07918	-0.07918	-0.07918	0.51816	0.51819	0.51818
1.00	-0.07174	-0.07175	-0.07175	0.49434	0.49437	0.49437
1.20	-0.05948	-0.05948	-0.05948	0.45069	0.45072	0.45072
1.40	-0.04982	-0.04983	-0.04983	0.41164	0.41166	0.41166
1.60	-0.04209	-0.04209	-0.04209	0.37651	0.37653	0.37653
1.80	-0.03580	-0.03580	-0.03580	0.34479	0.34481	0.34481
2.00	-0.03063	-0.03063	-0.03063	0.31606	0.31608	0.31608
2.50	-0.02117	-0.02118	-0.02118	0.25519	0.25521	0.25521
3.00	-0.01498	-0.01498	-0.01498	0.20689	0.20692	0.20692
3.50	-0.01078	-0.01078	-0.01078	0.16829	0.16832	0.16832
4.00	-0.00787	-0.00787	-0.00787	0.13727	0.13730	0.13730
4.50	-0.00581	-0.00582	-0.00582	0.11224	0.11226	0.11226
5.00	-0.00434	-0.00434	-0.00434	0.09197	0.09199	0.09199
6.00	-0.00247	-0.00247	-0.00247	0.06209	0.06211	0.06211
7.00	-0.00144	-0.00145	-0.00145	0.04218	0.04221	0.04220
8.00	-0.00086	-0.00086	-0.00086	0.02882	0.02884	0.02884
9.00	-0.00052	-0.00053	-0.00053	0.01978	0.01981	0.01981
10.00	-0.00032	-0.00032	-0.00032	0.01364	0.01367	0.01367

Table II. Values of Ω_1 and Θ_1 versus x.

N	8	9	10	8	9	10
x		Ω_1			Θ_1	
0.	0.34766	0.34769	0.34769	-0.44916	-0.44919	-0.44919
0.01	0.32989	0.32989	0.32989	-0.42945	-0.42945	-0.42945
0.02	0.31798	0.31798	0.31798	-0.41585	-0.41586	-0.41585
0.03	0.30810	0.30811	0.30811	-0.40442	-0.40442	-0.40442
0.04	0.29949	0.29949	0.29949	-0.39433	-0.39433	-0.39433
0.05	0.29176	0.29177	0.29177	-0.38520	-0.38521	-0.38521
0.10	0.26120	0.26121	0.26121	-0.34841	-0.34842	-0.34842
0.15	0.23834	0.23835	0.23835	-0.32021	-0.32022	-0.32022
0.20	0.21987	0.21987	0.21988	-0.29703	-0.29704	-0.29704
0.25	0.20434	0.20434	0.20435	-0.27728	-0.27728	-0.27728
0.30	0.19095	0.19096	0.19096	-0.26006	-0.26006	-0.26006
0.35	0.17921	0.17921	0.17921	-0.24480	-0.24481	-0.24481
0.40	0.16877	0.16877	0.16877	-0.23113	-0.23114	-0.23114
0.45	0.15939	0.15940	0.15940	-0.21877	-0.21878	-0.21878
0.50	0.15091	0.15092	0.15092	-0.20752	-0.20752	-0.20752
0.60	0.13610	0.13610	0.13611	-0.18770	-0.18770	-0.18770
0.70	0.12355	0.12356	0.12356	-0.17075	-0.17076	-0.17076
0.80	0.11275	0.11276	0.11276	-0.15606	-0.15607	-0.15607
0.90	0.10334	0.10335	0.10335	-0.14319	-0.14319	-0.14319
1.00	0.09506	0.09507	0.09507	-0.13180	-0.13181	-0.13181
1.20	0.08117	0.08118	0.08118	-0.11258	-0.11258	-0.11258
1.40	0.06998	0.06999	0.06999	-0.09700	-0.09700	-0.09700
1.60	0.06080	0.06080	0.06081	-0.08416	-0.08417	-0.08417
1.80	0.05315	0.05315	0.05315	-0.07344	-0.07345	-0.07345
2.00	0.04670	0.04671	0.04671	-0.06440	-0.06441	-0.06440
2.50	0.03442	0.03442	0.03443	-0.04717	-0.04717	-0.04717
3.00	0.02587	0.02588	0.02588	-0.03520	-0.03521	-0.03521
3.50	0.01973	0.01974	0.01974	-0.02666	-0.02666	-0.02666
4.00	0.01523	0.01524	0.01524	-0.02042	-0.02042	-0.02042
4.50	0.01186	0.01187	0.01187	-0.01579	-0.01579	-0.01579
5.00	0.00931	0.00932	0.00932	-0.01231	-0.01231	-0.01231
6.00	0.00585	0.00585	0.00585	-0.00762	-0.00763	-0.00763
7.00	0.00374	0.00375	0.00375	-0.00482	-0.00483	-0.00483
8.00	0.00243	0.00244	0.00244	-0.00310	-0.00310	-0.00310
9.00	0.00160	0.00161	0.00161	-0.00202	-0.00203	-0.00203
10.00	0.00106	0.00107	0.00107	-0.00133	-0.00134	-0.00134

Table III. Values of the constants $\varkappa_1$, α_1, $\varkappa_2^{(2)}$, $\alpha_2^{(2)}$ and $\alpha_2^{(3)}$.

N	8	9	10
$\varkappa_1$	1.0161900	1.0161919	1.0161921
α_1	1.3027112	1.3027163	1.3027159
$\varkappa_2^{(2)}$	-1.1537188	-1.1537433	-1.1537431
$\alpha_2^{(2)}$	1.1617220	1.1617245	1.1617237
$\alpha_2^{(3)}$	-1.715690	-1.716021	-1.716073

Table IV. Values of $\Omega_2^{(2)}$ and $\Theta_2^{(2)}$ versus x.

N / x	8 $5\cdot\Omega_2^{(2)}$	9 $5\cdot\Omega_2^{(2)}$	10 $5\cdot\Omega_2^{(2)}$	8 $\Theta_2^{(2)}$	9 $\Theta_2^{(2)}$	10 $\Theta_2^{(2)}$
0.	0.46102	0.46121	0.46121	-0.44901	-0.44901	-0.44902
0.01	0.41788	0.41790	0.41791	-0.43753	-0.43753	-0.43753
0.02	0.38859	0.38861	0.38861	-0.42845	-0.42845	-0.42845
0.03	0.36427	0.36429	0.36430	-0.42035	-0.42035	-0.42035
0.04	0.34306	0.34309	0.34310	-0.41290	-0.41290	-0.41290
0.05	0.32409	0.32413	0.32413	-0.40595	-0.40595	-0.40595
0.10	0.24988	0.24995	0.24995	-0.37616	-0.37617	-0.37617
0.15	0.19585	0.19591	0.19592	-0.35169	-0.35169	-0.35169
0.20	0.15362	0.15367	0.15367	-0.33066	-0.33066	-0.33066
0.25	0.11940	0.11945	0.11945	-0.31213	-0.31213	-0.31213
0.30	0.09106	0.09110	0.09110	-0.29556	-0.29556	-0.29556
0.35	0.06724	0.06728	0.06728	-0.28057	-0.28057	-0.28057
0.40	0.04702	0.04706	0.04706	-0.26690	-0.26691	-0.26691
0.45	0.02972	0.02976	0.02976	-0.25436	-0.25436	-0.25436
0.50	0.01484	0.01488	0.01489	-0.24277	-0.24278	-0.24278
0.60	-0.00912	-0.00907	-0.00907	-0.22205	-0.22205	-0.22205
0.70	-0.02713	-0.02707	-0.02707	-0.20399	-0.20399	-0.20399
0.80	-0.04067	-0.04062	-0.04061	-0.18808	-0.18808	-0.18808
0.90	-0.05081	-0.05076	-0.05076	-0.17395	-0.17396	-0.17395
1.00	-0.05831	-0.05826	-0.05825	-0.16132	-0.16132	-0.16132
1.20	-0.06751	-0.06747	-0.06746	-0.13967	-0.13967	-0.13967
1.40	-0.07143	-0.07138	-0.07138	-0.12183	-0.12183	-0.12183
1.60	-0.07197	-0.07193	-0.07193	-0.10692	-0.10692	-0.10692
1.80	-0.07037	-0.07033	-0.07033	-0.09431	-0.09431	-0.09431
2.00	-0.06744	-0.06740	-0.06740	-0.08355	-0.08356	-0.08356
2.50	-0.05741	-0.05736	-0.05736	-0.06269	-0.06270	-0.06269
3.00	-0.04662	-0.04657	-0.04656	-0.04787	-0.04787	-0.04787
3.50	-0.03680	-0.03675	-0.03675	-0.03705	-0.03705	-0.03705
4.00	-0.02848	-0.02843	-0.02843	-0.02898	-0.02898	-0.02898
4.50	-0.02170	-0.02165	-0.02165	-0.02287	-0.02287	-0.02287
5.00	-0.01630	-0.01626	-0.01626	-0.01818	-0.01818	-0.01818
6.00	-0.00882	-0.00878	-0.00878	-0.01169	-0.01169	-0.01169
7.00	-0.00445	-0.00441	-0.00440	-0.00766	-0.00766	-0.00766
8.00	-0.00200	-0.00195	-0.00194	-0.00509	-0.00509	-0.00509
9.00	-0.00068	-0.00062	-0.00062	-0.00342	-0.00342	-0.00342
10.00	-0.00002	0.00004	0.00004	-0.00232	-0.00232	-0.00232

Table V. Values of $\Omega_2^{(3)}$ and $\Theta_2^{(3)}$ versus x.

N / x	8	9 ($\Omega_2^{(3)}$)	10	8	9 ($\Theta_2^{(3)}$)	10
0.	-1.62543	-1.62828	-1.62841	0.69351	0.69486	0.69482
0.01	-1.61148	-1.61194	-1.61209	0.70820	0.70839	0.70846
0.02	-1.60025	-1.60057	-1.60070	0.71570	0.71582	0.71589
0.03	-1.59030	-1.59074	-1.59086	0.72100	0.72118	0.72124
0.04	-1.58121	-1.58180	-1.58192	0.72504	0.72529	0.72534
0.05	-1.57276	-1.57348	-1.57360	0.72823	0.72854	0.72859
0.10	-1.53613	-1.53711	-1.53725	0.73683	0.73728	0.73732
0.15	-1.50457	-1.50547	-1.50562	0.73883	0.73923	0.73928
0.20	-1.47562	-1.47637	-1.47652	0.73727	0.73760	0.73766
0.25	-1.44830	-1.44894	-1.44908	0.73346	0.73373	0.73379
0.30	-1.42212	-1.42272	-1.42285	0.72809	0.72835	0.72841
0.35	-1.39684	-1.39743	-1.39756	0.72163	0.72188	0.72194
0.40	-1.37229	-1.37292	-1.37304	0.71436	0.71463	0.71468
0.45	-1.34839	-1.34906	-1.34919	0.70649	0.70678	0.70683
0.50	-1.32507	-1.32579	-1.32591	0.69818	0.69850	0.69854
0.60	-1.27996	-1.28077	-1.28090	0.68064	0.68100	0.68105
0.70	-1.23669	-1.23755	-1.23769	0.66237	0.66276	0.66281
0.80	-1.19508	-1.19596	-1.19610	0.64377	0.64416	0.64421
0.90	-1.15500	-1.15586	-1.15601	0.62508	0.62546	0.62552
1.00	-1.11634	-1.11717	-1.11732	0.60646	0.60684	0.60689
1.20	-1.04299	-1.04373	-1.04388	0.56991	0.57024	0.57030
1.40	-0.97452	-0.97519	-0.97533	0.53468	0.53497	0.53502
1.60	-0.91054	-0.91118	-0.91131	0.50103	0.50131	0.50136
1.80	-0.85074	-0.85139	-0.85151	0.46911	0.46939	0.46944
2.00	-0.79485	-0.79553	-0.79565	0.43895	0.43925	0.43929
2.50	-0.67067	-0.67147	-0.67160	0.37113	0.37149	0.37153
3.00	-0.56601	-0.56688	-0.56702	0.31339	0.31378	0.31383
3.50	-0.47789	-0.47874	-0.47889	0.26453	0.26491	0.26497
4.00	-0.40369	-0.40448	-0.40463	0.22329	0.22364	0.22370
4.50	-0.34119	-0.34191	-0.34205	0.18852	0.18884	0.18890
5.00	-0.28851	-0.28916	-0.28930	0.15922	0.15951	0.15956
6.00	-0.20655	-0.20717	-0.20729	0.11369	0.11396	0.11400
7.00	-0.14804	-0.14874	-0.14886	0.08127	0.08157	0.08162
8.00	-0.10619	-0.10700	-0.10713	0.05815	0.05850	0.05855
9.00	-0.07621	-0.07711	-0.07726	0.04164	0.04203	0.04209
10.00	-0.05472	-0.05566	-0.05582	0.02983	0.03025	0.03031

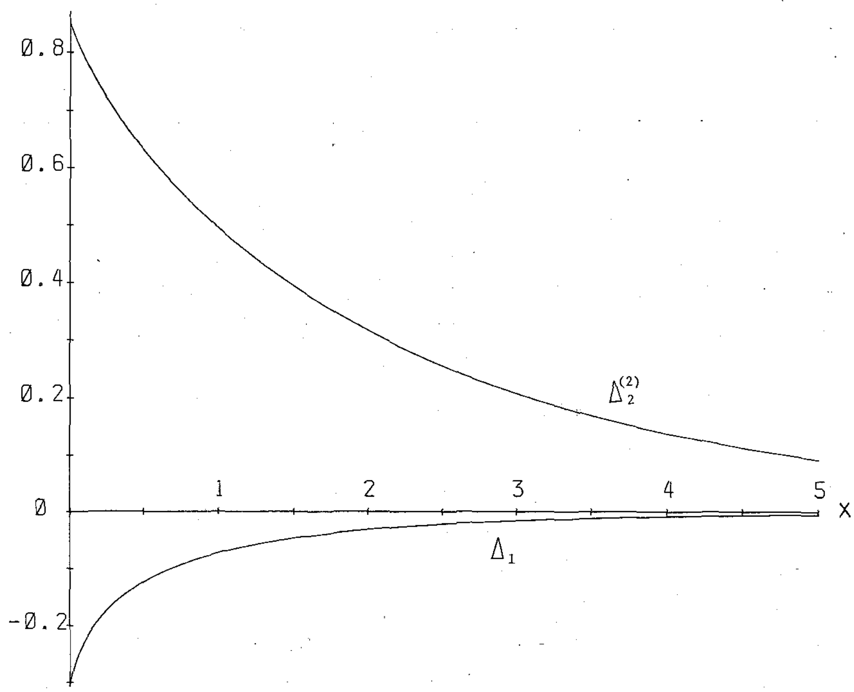

Fig.1. Plots of Δ_1 and $\Delta_2^{(2)}$.

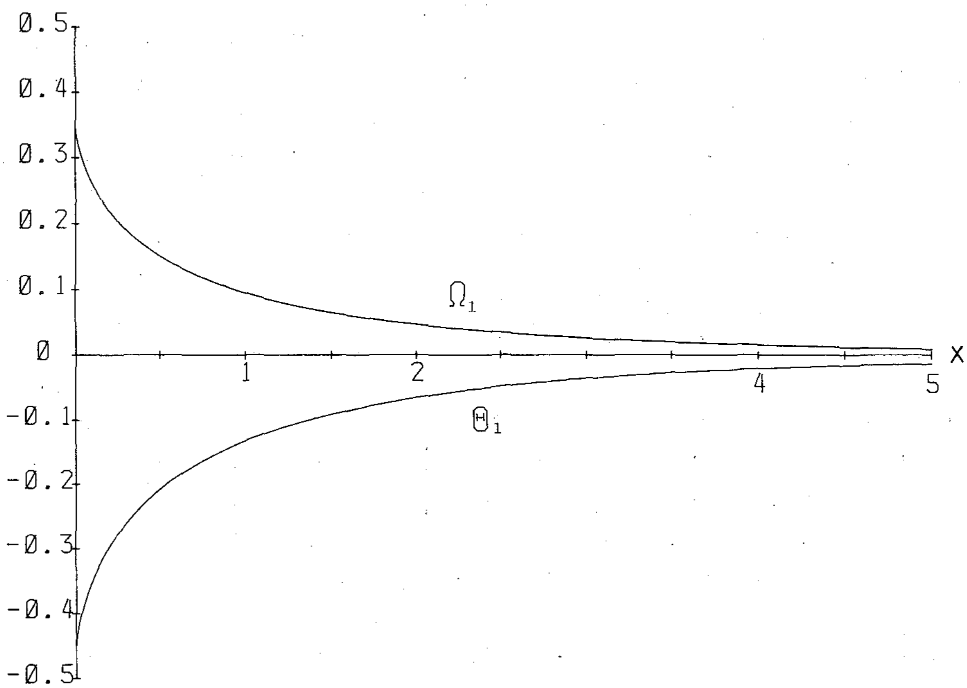

Fig.2. Plots of Ω_1 and Θ_1.

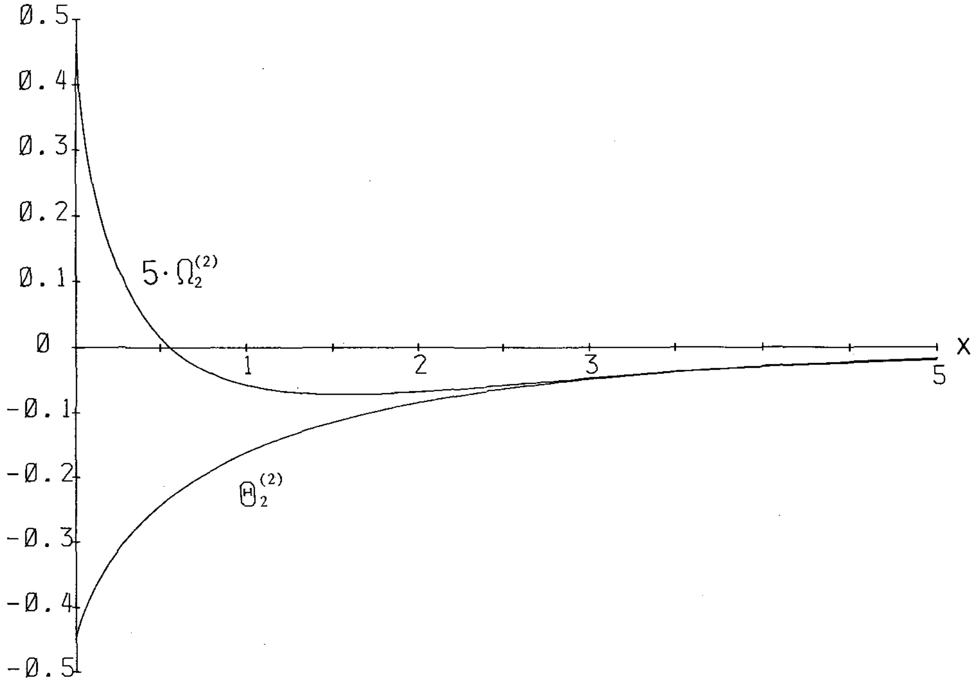

Fig.3. Plots of $\Omega_2^{(2)}$ and $\Theta_2^{(2)}$.

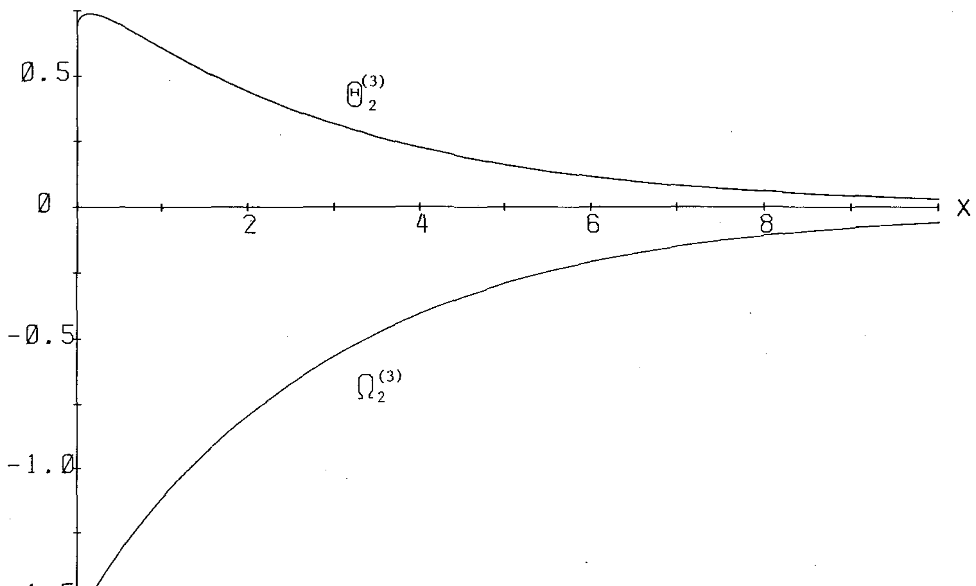

Fig.4. Plots of $\Omega_2^{(3)}$ and $\Theta_2^{(3)}$.