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System Identification Using Fast Fourier Transform

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Synopsis

Algorithms for system identification applying throughout Fast Fourier Transform (FFT) to the major calculating operations are introduced.

It is shown that by using data of about as twice length as system settling time and by truncating the incorrect correlation functions resulting from them, errors owing to finiteness of data can be avoided. It is shown that so as to suppress the effects owing to statistical fluctuation of input data or output noise, superposition of data in frequency domain is effective, and also the damping terms of poles or zeros can be efficiently evaluated by utilizing the phase change of the spectra of the impulse response sequence.

The proposed method can be efficiently applied to relatively higher order systems or relatively rapidly time-variant systems because of high accuracy and high speed processing of FFT. Moreover, it needs not to assume the order of the system a priori, and yields a reasonable lower order approximating system in itself.

I. Introduction

In the problem identifying the impulse response of a linear system , it is well known that correlation methods or least square error methods have advantages that a random signal can be utilized as an

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identification signal and the effect of noise is to be suppressed considerably. In the higher order system, however, they have disadvantage in the data processing.

In these cases, Fast Fourier Transform (FFT) [1,3] seems to be a strong weapon. But, it seems to be used very little in system identification.

Recently, M.J.Corinthios [1] has designed a special-purpose machine for FFT and has shown that FFT can be effectively utilized for system identification as an application of his machine. It is the identification method which searches poles and zeros of an unknown linear system when its impulse response sequence are given. Identification of the damping terms of poles or zeros, however, are unsatisfactorily dealt with. Moreover, the cases are scarcely discussed which identification input signals are not impulsive signals.

In this paper, the following two-stage identification method is proposed which uses FFT in almost major data processing. And some difficulties in which yield are discussed.

First stage: Estimation of the impulse response with a correlation method using the input-output data of the unknown system which has a random signal as an input signal.

Second stage: Estimation of the poles and the zeros using the impulse response sequence.

And some examples are illustrated with computer simulations in order to prove the effectiveness of the proposed method.

II. Problem and an Outline of the Approach

Generally, the input-output relation of the discrete-time linear system which is stable, stationary and has an observation noise, is approximately represented as the following (1) using the finite impulse response $\{f_k\}$ $k \in \underline{N}$. (c.f. Fig. 1)

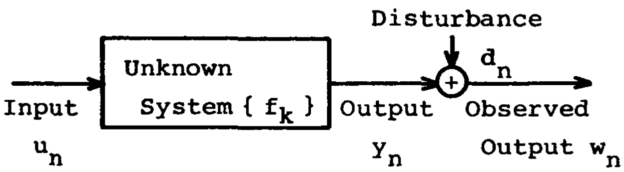

Fig. 1 Discrete-time Linear System

$$\begin{cases} y_n = \sum_{k=0}^{N-1} f_k \cdot u_{n-k} & ; n = 0, 1, 2, \ldots \quad (1.a) \\ w_n = y_n + d_n & ; n = 0, 1, 2, \ldots \quad (1.b) \\ \mathbf{f}_N = (f_0, f_1, \ldots, f_{N-1})^T & \quad (1.c) \end{cases}$$

where u_n, y_n, w_n and d_n are the n-th time input, output, observed output and observation noise, respectively. And $\underline{N}$ represents the set $\{0, 1, 2, \ldots, N-1\}$, and $*^T$ is the transpose of $*$.

Let the pulse transfer function F(z) of the system (1) be (2.a) and let the poles' set Λ_p and the zeros' set Λ_q be (2.b) and (2.c), respectively.

$$F(z) = \left\{ \prod_{j=0}^{m-1} (1 - q_j \cdot z^{-1}) \right\} / \left\{ \prod_{i=0}^{n-1} (1 - p_i \cdot z^{-1}) \right\} \tag{2.a}$$

$$\Lambda_p \triangleq \{ p_0, p_1, \ldots, p_{n-1} \} \tag{2.b}$$

$$\Lambda_q \triangleq \{ q_0, q_1, \ldots, q_{m-1} \} \tag{2.c}$$

It is the purpose of this paper to throw light upon the difficulties and the effectiveness of the case which introduces FFT as an algorithm estimating the finite approximate impulse response $\mathbf{f}_N$ and identifying the poles' set Λ_p and the zeros' set Λ_q of the pulse transfer function F(z).

The merit of the proposed method which first, estimates $\mathbf{f}_N$ and next, identifies Λ_p and Λ_q of F(z) using FFT algorithm, is 1) to be needless to assume the order of the unknown system a priori, 2) to estimate the some principal parts out of Λ_p and Λ_q in accordance with the reliability of the input/output data, and 3) to obtain a high accuracy estimate for even a considerably high order system with short time for the sake of simplified computational operation owing to FFT.

Whereas, in the operations with FFT, it is indispensable to devise so as to avoid the errors resulting from cyclic data processing.

In the following sections, the above problem is discussed in the first stage and the second stage, separately.

III. Estimation in the First Stage
— Estimation of $\mathbf{f}_N$ from input/output data —

III-1 Foundation of Estimation

In the estimation of the impulse response $\mathbf{f}_N$, the correlation technique is effective when the observation noise and the edge effects owing to the finiteness of data are taken into consideration.

Now, when (1) is valid with sufficient accuracy, the output sequence influenced by $\{u_k\}$ $k \in \underline{N}$ is only $\{y_k\}$ $k \in \underline{2N}$. Therefore, the cross-correlation function between the input and the output can be calculated as follows. [3] By the way, the auto-correlation function of the input can also be calculated in the quite same manner.

$$r_k = \frac{1}{N}\cdot {}_k\mathbf{u}_{N0}{}^T \cdot \mathbf{y}_{2N} \qquad ;\ k \in \underline{N} \tag{3}$$

$$\mathbf{r}_N \triangleq (\ r_0,\ r_1,\ .\ .\ .,\ r_{N-1}\ )^T \tag{4}$$

where

$$\mathbf{u}_{N0} \triangleq (\ u_0,\ u_1,\ .\ .\ .\ .,\ u_{N-1},\ 0,\ .\ .\ .\ .,\ 0\ )^T \tag{5}$$

$$\mathbf{y}_{2N} \triangleq (\ y_0,\ y_1,\ .\ .\ .\ .,\ y_{N-1},\ y_N,\ .\ .\ .,\ y_{2N-1}\ )^T \tag{6}$$

And, ${}_k\mathbf{u}_{N0}$ is what are carried out the k-times replacements for the each element of $\mathbf{u}_{N0}$.

As (3) requires a number of calculating operations, it is necessary to devise in order to save the number of operations.

Let's explain the method using FFT as a device for the sake of it.

Now, (3) is represented as (7) in the vector form.

$$\tilde{\mathbf{r}}_{2N} = \frac{1}{N}\cdot\Gamma\cdot y_{2N} \tag{7}$$

where

$$\tilde{\mathbf{r}}_{2N} \triangleq (\ r_0,\ r_1,\ .\ .\ .\ ,\ r_{N-1},\ r_N,\ \tilde{r}_{N+1},\ .\ .\ .,\ \tilde{r}_{2N-1}\ )^T \tag{8}$$

$$\Gamma \triangleq (\ {}_0\mathbf{u}_{N0},\ {}_1\mathbf{u}_{N0},\ .\ .\ .\ .,\ {}_{2N-1}\mathbf{u}_{N0}\ )^T \tag{9}$$

And, since all the later than n-th element of $\mathbf{u}_{2N}$ have the errors owing to cyclic data processing, these elements can not be utilized so as to estimate $\mathbf{f}_N$. Hereafter, the mark '$\sim$' is put to the variables which involve these kind of errors.

Let the Discrete Fourier Transform (DFT) matrix with 2N data points be D_{2N}. And so, DFT of (7) is expressed as (10).

$$D_{2N}\cdot\tilde{\mathbf{r}}_{2N} = \frac{1}{N}\, D_{2N}\cdot\Gamma\cdot\mathbf{y}_{2N} \tag{10}$$

where

$$D_{2N} \triangleq \begin{pmatrix} 1 & 1 & 1 & .\ .\ .\ . & 1 \\ 1 & w^1 & w^2 & .\ .\ .\ . & w^{2N-1} \\ . & . & . & .\ .\ .\ . & . \\ 1 & w^{2N-1} & w^{2N-2} & .\ .\ .\ . & w^1 \end{pmatrix} \tag{11}$$

$$w \triangleq \exp(\ i\cdot 2\pi/2N\ ) \quad ; \quad i \triangleq \sqrt{-1}$$

Well, the relation (12) is valid. (Proof in Appendix I)

$$\Gamma\cdot D_{2N}{}^{-1} = D_{2N}{}^{-1}\cdot\Lambda(U_0{}^*) \tag{12}$$

Let DFT of $\tilde{\mathbf{r}}_{2N}$ and $\mathbf{y}_{2N}$ represent $\tilde{\mathbb{R}}_{2N}$ and $\mathbf{Y}_{2N}$, respectively. And then, (10) is rewritten as (13).

$$\tilde{\mathbb{R}}_{2N} = \frac{1}{N}\cdot\Lambda(U_0{}^*)\cdot\mathbf{Y}_{2N} \tag{13}$$

where

$$\Lambda(U_0{}^*) \triangleq \mathrm{diag}(\ U_{00}{}^*,\ U_{01}{}^*,\ .\ .\ .\ .\ .,\ U_{02N-1}{}^*) \tag{14}$$

$$\mathbf{U}_0{}^* \triangleq (\ D_{2N}\,\mathbf{u}_{N0}\ )^* = (\ U_{00}{}^*,\ U_{01}{}^*,\ .\ .\ .\ .\ .,\ U_{02N-1}{}^*\ )^T \tag{15}$$

And, $(\ )^*$ represents the complex conjugate of ( ).

Well, on account of the later than N-th element of $\tilde{\mathbf{r}}_{2N}$, all the elements of $\tilde{\mathbb{R}}_{2N}$ have the error of the meaning '$\sim$'. Therefore, $\tilde{\mathbb{R}}_{2n}$ can not be utilized for the purpose of estimting f_N if they were still what they are. In order to obtain the right DFT $\mathbb{R}_N$ of $\mathbf{r}_N$ in (4), it is necessary for the data processing as (16) to be carried out.

$$\mathbb{R}_N = D_N\cdot\mathrm{trunc}\left\{ D_{2N}{}^{-1}\ \tilde{\mathbb{R}}_{2N}\right\}_N \tag{16}$$

where, $\mathrm{trunc}\{*\}_N$ represents the first N terms of $*$.

From the above discussion, it is shown that the calculation of R_N in (16) requires only twice DFT and once IDFT (Inverse DFT) with 2N data points and once DFT with N data points.

In the same manner, DFT $\mathbf{Q}_N$ of the auto-correlation function vector $\mathbf{q}_N$ of the input can be obtained in (17).

$$\mathbf{Q}_N = D_N\cdot\mathrm{trunc}\left\{ D_{2N}{}^{-1}\ \tilde{\mathbf{Q}}_{2N}\right\}_N \tag{17}$$

where

$$\tilde{\mathbf{Q}}_{2N} \triangleq D_{2N}\cdot\mathbf{q}_{2N} = \frac{1}{N}\cdot\Lambda(U_0{}^*)\cdot U_{2N} \tag{18}$$

$$\mathbf{U}_{2N} \triangleq D_{2N}\cdot\mathbf{u}_{2N}$$

$$\mathbf{u}_{2N} \triangleq (\ u_0,\ u_1,\ .\ .\ .\ .,\ u_{N-1},\ u_N,\ .\ .\ .\ .,\ u_{2N-1}\ )^T \tag{19}$$

From $\mathbb{R}_N$ and $\mathbf{Q}_N$, DFT $\mathbb{F}_N$ of $\mathbf{f}_N$ is obtained in the form (20) which is the division of each element of $\mathbb{R}_N$ and $\mathbf{Q}_N$.

$$F_k = R_k/Q_k \qquad ;\ k\in \underline{N} \tag{20}$$

$$\mathbb{F}_N \triangleq (\ F_0,\ F_1,\ .\ .\ .\ .,\ F_{N-1}\ )^T \tag{21}$$

In the practical estimation of $\mathbb{F}_N$, it is necessary that $Q_k \neq 0$; $k\in\underline{N}$ are valid with sufficient margine, respectively and that the effect of the observation noise is diminished.

These will be discussed in the following section.

Let us give a definition that we shall use in the following.

< Definition >

When for a suitable α ($0<\alpha<1$) and Q_N in (17), (22) is valid, we call the input sequence $\{u_k\}k\in\underline{2N}$ to be practically sufficiently general (PSG).

$$\min_{k\in\underline{N}} |Q_k| \geqq \alpha\cdot\left\{\operatorname*{mean}_{k\in\underline{N}} |Q_k|\right\} \tag{22}$$

III-2 Practical Estimation

The practical estimation method is almost same as the estimation foundation in III-1 as the outline, except that a suitable superposition of data is required in order to satisfy the condition PSG and to diminish the effect of the observation noise.

The superposition of data in the step of $\tilde{\mathbb{R}}_{2N}$ and $\tilde{\mathbb{Q}}_{2N}$ in (13) and (18) is most effective on the points of the number of calculating operations and the effect of the superposition.

Well, the procedure is explained as follows.

First, to calculate $\bar{\tilde{\mathbb{R}}}_{2N}$ and $\bar{\tilde{\mathbb{Q}}}_{2N}$ in $(13)^*$ and $(18)^*$, respectively.

Second, to obtain $\mathbb{R}_N$ and $\mathbb{Q}_N$ regarding $\bar{\tilde{\mathbb{R}}}_{2N}$ and $\bar{\tilde{\mathbb{Q}}}_{2N}$ as $\tilde{\mathbb{R}}_{2N}$ and $\tilde{\mathbb{Q}}_{2N}$ in (16) and (17), respectively. And last, to estimate $\mathbb{F}_N$ in (20).

$$\bar{\tilde{\mathbb{R}}}_{2N} = \frac{1}{M}\cdot\sum_{m=0}^{M-1} \frac{1}{N}\cdot\Lambda(\,{}_mU_0^{\,*})\cdot {}_m\mathbb{W}_{2N} \tag{13*}$$

$$\bar{\tilde{\mathbb{Q}}}_{2N} = \frac{1}{M}\cdot\sum_{m=0}^{M-1} \frac{1}{N}\cdot\Lambda(\,{}_mU_0^{\,*})\cdot {}_m\mathbb{U}_{2N} \tag{18}$$

where

$${}_m\mathbb{W}_{2N} \triangleq D_{2N}\cdot(\,{}_m\mathbb{y}_{2N} + {}_m\mathbb{d}_{2N}\,)\quad,$$

M represents the number of the superposition.

Such effects of the superposition for PSG and the observation noise shall be discussed in the following.

III-3 The effect of the superposition

Let a gaussian white noise sequence with zero mean be $\{u_k\}^m\, k\in\underline{N}$, $m\in\underline{M}$ and let the DFT of them be $\{U_k^m\}$, $k\in\underline{N}$, $m\in\underline{M}$ (M : the number of the superposition). Moreover, let the real part and the imaginary part of U_k^m be x_{km} and y_{km} ($k \in \underline{N}$, $m \in \underline{M}$), respectively.

Since the data processing of DFT is a linear transformation, these $\{x_{km}, y_{km}\}$ $k \in \underline{N}$, $m \in \underline{M}$ can be considered as random variables being subject to a gaussian distribution $N(0, \sigma^2)$ assuming that $\{u_k\}$ is a gaussian sequence with zero mean as the above assumption.

Now, let us consider the following three kinds of superposition of $\{U_k^m\}$.

i) the superposition in the form of the absolute value of U_k^m

ii) the superposition in the form of the sqare of the absolute value of U_k^m

iii) the superposition in the form of the complex variable U_k^m in itself

The statistical properties are shown in Table 1.

Table 1. The comparison among three kinds of superposition in their statistical properties

kinds of superposition	mean	variance
i) $\frac{1}{M}\sum_{m=0}^{M-1}\lvert x_{km} + jy_{km}\rvert$	$(\sqrt{\pi}/2)\cdot\sigma^2$	$(2/M)\cdot\sigma^2$
ii) $\frac{1}{M}\sum_{m=0}^{M-1}\lvert x_{km} + jy_{km}\rvert^2$	$2\cdot\sigma^2$	$(4\cdot\sigma^4)/M$
iii) $\lvert\frac{1}{M}\sum_{m=0}^{M-1}(x_{km} + jy_{km})\rvert$	$(\sqrt{\pi}/2M)\cdot\sigma^2$	$(2/M)\cdot\sigma^2$

where $U_k^m = x_{km} + jy_{km}$; k $\underline{N}$, m $\underline{M}$, x_{km}, y_{km} $N(0, \sigma^2)$

From this table, it is shown that in both i) and ii), the mean value is constant with respect to M and only the variance decreases in inverse proportion to M, whereas, in iii), both the mean and the variance decrease in inverse proportion to M as M increases.

The method of the superposition discussed in III-2 is regarded as the superposition of ii) and iii), because both the first term of $(13)^*$ and $(18)^*$ are approximately regarded as the superposition of ii), and both the real part and the imaginary part of $(1/N)\cdot\Lambda({}_mU_0^*)\cdot{}_mD_{2N}$ in the second term of $(13)^*$ are considered as a gaussian sequence with zero mean and a finite variance assuming that $\{u_k\}$ and $\{d_k\}$ are mutually independent. Therefore, it is obvious that in this case, the effect of the observation noise decreases in inverse proportion to M.

Moreover, since (18)* is approximately regarded as the superposition of ii), the condition PSG is effectively improved.

On the other hand, in the method which the direct superposition of U_k and Y_k are carried out, both the effect of the observation noise and the condition PSG are scarcely improved, because their superposition are regarded as those of iii).

IV. The Second Stage Estimation
— The Estimation of Λ_p , Λ_q —

IV-1 The Foundation of the Identification

The foundation of the identification is to utilize the fact that the absolute value of the pulse transfer function F(z) has a positive (negative) directional peak value at a pole (zero) of F(z) in the z-plane. In the following, so as to simplify the description, let us discuss only the estimation of some poles of Λ_p.

Now, let the z-transform of the IDFT $\mathbf{f}_N$ of $\mathbf{F}_N$ estimated in the first stage be $F_N(z)$ in (23).

$$F_N(z) \triangleq \sum_{k=0}^{N-1} f_k \cdot z^{-k} \tag{23}$$

where, $\mathbf{f}_N \triangleq (f_0, f_1, \ . \ . \ . \ , f_{N-1})^T$.

The relation between F(z) in (2) and $F_N(z)$ in (23) can be described as follows.

$$F(z_k) \cong F_N(z_k) \tag{24}$$

where, $z_k \triangleq \exp(\sigma + i{\cdot}k{\cdot}2\pi/N) \ ; \quad \sigma \leqq 0 \ , \ k \in \underline{N}$. (25)

In other words, there is the relation that $F_N(z)$ does approximately equal to F(z) only at z_k defined in (25). Therefore, it is possible to estimate any element of Λ_p or Λ_q using $F_N(z)$ with accuracy of z_k (frequency resolution : $2\pi/N$).

As σ in (25) diminishes (i.e., σ is negative and the absolute value of σ increases), however, the accuracy of (24) becomes poor and poor. Therefore, it is necessary to begin with the estimation of the pole (zero) whose σ is nearest to zero in the poles (zeros) corresponding to the sequence $\{f_k\}$ $k \in \underline{N}$, and to eliminate the estimated pole (zero) from the sequence $\{f_k\}$ $k \in \underline{N}$ in turns.

And also, let z_k in the case $\sigma = 0$ be z_{0k} ($k \in \underline{N}$). Then, there is the relation that $F_N(z_{0k}) = F_k$ (where F_k is the k-th element of $\mathbb{F}_N$).

IV-2 The Practical Identification

When we evaluate the value of $F_N(z)$ at z_k of (25) in the z-plane, it is convenient for us to investigate it along both a constant damping contour formed in a concentric circle and a constant frequency contour formed in a radial line segment as Fig.2.

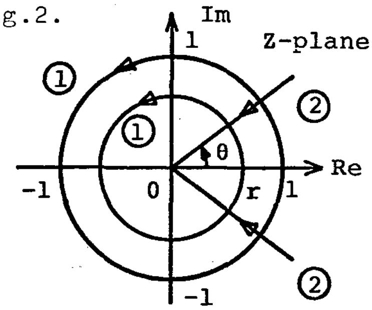

Fig. 2 Contour on Z-plane

In the following, when a pole p is represented as $p = r_p \cdot \exp(i \cdot \theta_p)$, we call the estimate of θ_p one of the phase term and the estimation of r_p one of the damping term, respectively.

In the contour ①, we have only to calculate (26). Since this is the DFT of $\{g_k\}$ $k \in \underline{N}$, the number of calculations equals to one of once DFT with N data points. And from this spectrum, it is possible to estimate the phase term θ_p . (frequency resolution : $2\pi/N$)

$$F_N(z_k) = \sum_{k=0}^{N-1} f_k \cdot z_j^{-k} = \sum_{k=0}^{N-1} g_k \cdot z_{0j}^{-k} \quad ; \; j \in \underline{N} \tag{26}$$

where,

$$g_k \triangleq f_k \cdot \exp(-k \cdot \sigma) \quad ; \; k \in \underline{N} \;,\; \sigma = \text{const.} \tag{27}$$

$$z_{0j} \triangleq \exp(\, i \cdot j \cdot 2\pi/N \,) \;,\; j \in \underline{N} \;,\; i \triangleq \sqrt{-1} \;. \tag{28}$$

On the other hand, in the contour ② which is necessary in order to estimate the damping term r_p , there is no peak in the spectrum $F_N(z_{\sigma j})$, because the calculations of (23) at $z_{\sigma j}$ in (29) are those of summation over the terms multiplied by only real values, respectively.

$$z_{\sigma j} = \exp(\sigma_j + i \cdot m \cdot 2\pi/N \,) \;;\; m = \text{const.},\; j \in \underline{L} \tag{29}$$

(L : arbitrary, i.e., independent of N)

Therefore, it is necessary to make use of the phase information in order to estimate the damping term.

Let us discuss the estimation of the damping term using the phase change in two special cases as follows.

i) The case which a pole is sufficiently separated from other poles and (30) is approximately valid.

$$F_N(z) = \Sigma_1(z)\cdot\sum_{k=0}^{N-1} (p\cdot z^{-1})^k \tag{30}$$

where $\Sigma_1(z)$ is the contribution from poles and zeros except for the pole p.

Moreover, let the phase of $F_N(z)$ be $\varphi(z)$ and generally, let the phase of A represent arg $\{A\}$. Then, $\varphi(z)$ is written as (31).

$$\varphi(z) = \arg\{\Sigma_1(z)\} + \arg\left\{\sum_{k=0}^{N-1} (p\cdot z^{-1})^k\right\} \tag{31}$$

In this case, the following theorem can be obtained. (The proof is given in Appendix II.)

〈Theorem 1〉

Assuming that (30) is valid, when we compute the change of $\varphi(z)$ along the contour ① near the pole p, (32) is obtained.

$$\left.\frac{d\varphi(z)}{d\theta}\right|_{\theta=\theta_p} < 0 \quad ; \quad r > r_p$$
$$\left.\frac{d\ \ (z)}{d}\right|_{\theta=\theta_p} > 0 \quad ; \quad r < r_p \tag{32}$$

where, $z \triangleq r\cdot\exp(i\cdot\theta)$, $p \triangleq r_p\cdot\exp(i\cdot\theta_p)$.

According to this theorem, we can exactly estimate the damping term r_p in theory.

ii) The case which two adjacent poles are on the same phase (frequency resolution : $2\pi/N$) with being sufficiently separated from other poles and (33) is approximately valid.

$$F_N(z) = \Sigma_2(z)\cdot\left\{\sum_{k=0}^{N-1} \left(\frac{1}{p_1-p_2} \right)\cdot(p_1^{k+1} - p_2^{k+1})\cdot z^{-k}\right\} \tag{33}$$

By the discussion similar to the case i), the following theorem is obtained. (The proof is given in Appendix III.)

〈Theorem 2〉

Assuming that (33) is valid, when we compute the change of $\varphi(z)$ along the contour ① near the two poles p_1 and p_2, (34) is obtained.

$$\left.\frac{d\varphi(z)}{d\theta}\right|_{\theta=\theta_p} < 0 \quad ; \quad r > r_1 > r_2$$

$$\left.\frac{d\,\varphi(z)}{d\,\theta}\right|_{\theta=\theta_p} > 0 \quad ; \quad r_1 > r > r_2 \tag{34}$$

$$\left.\frac{d\quad(z)}{d}\right|_{\;=\;\;p} < 0 \quad ; \quad r_1 > r_2 > r$$

where, $z \triangleq r\cdot\exp(i\cdot\theta)$, $p_j = r_j\cdot\exp(i\cdot\theta_p)$; $j=1,2$, $r_1 > r_2$.

Therefore, the procedure is necessary with which we begin with the estimation of the most outside pole and after eliminating it from the sequence $\{f_k\}\,k\in\underline{N}$, we estimate the next outside pole again.

Well, though we don't think that by considering only the above two cases, all the cases have been investigated out or possible to be identified, we think that any more complicated judgement is needless considering $2\pi/N$ frequency resolution.

IV-3 Identification Algorithms

Identification algorithms through the first stage and the second stage discussed in the above sections are represented as Fig.3.

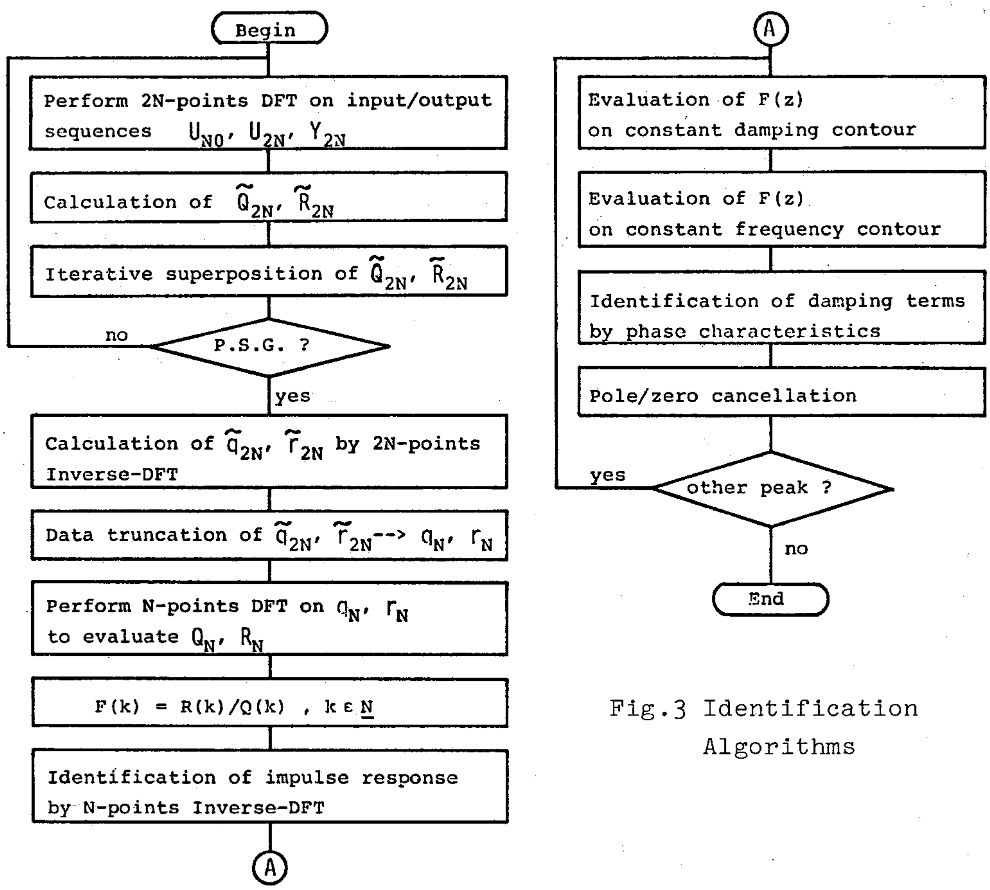

Fig.3 Identification Algorithms

V. Computer Simulations

As an illustration the amplitude and phase of constant-damping transforms and impulse responses obtained from input/output data for the following two systems are shown in Fig.4 - 6.

(Ex-1) The case of the third order system whose the pulse transfer function is (35).

$$F(z) = z^{-3}/\{(1 - p_1 z^{-1})\cdot(1 - p_2 z^{-1})\cdot(1 - p_2^* z^{-1})\} \tag{35}$$

$$p_1 = 0.980 \ , \ p_2 = 0.960\cdot\exp(\ i\cdot\pi/4\ )$$

(Ex-2) The case of the ninth order system whose the pulse transfer function is (36).

$$F(z) = \left\{\prod_{j=1}^{6} (1 - q_j z^{-1})\right\}/\left\{\prod_{i=1}^{9} (1 - p_i z^{-1})\right\} \tag{36}$$

$$\{p_i\} = \{0.980,\ (0.970,\ 45°),\ (0.950,\ 30°),\ (0.940,\ 60°),\ (0.920,\ 80°)\}$$

$$\{q_j\} = \{(0.930,\ 35°),\ (0.900,\ 50°),\ (0.830,\ 70°)\}$$

where, $(r_j,\ \theta_j°)$ represents $r_j\cdot\exp(\pm i\cdot\theta_j°)$.

As an identification input signal a pseudo gaussian random sequence is used, and as an observation noise another pseudo gaussian random sequence being independent of the input sequence is added to the output. The ratio of the noise to the input in standard deviation is set as 0.1. And it is set that the number of sample points 2N = 512 (or 2N = 256) and the number of superposition M = 5 (or M = 10).

Fig.4 shows the impulse responses resulting from the finite approximate pulse transfer function estimated in the first stage identification. Fig.4 (a) and (b) are one of the case S/N ratio = 0.0 and one of the case S/N ratio = 0.1 for the third order system (Ex-1), respectively. And Fig.4 (c) and (d) are those for the ninth order system (Ex-1), respectively.

These figures show that at about (2N/3)-th sample time, impulse response sequences are sufficiently settling, and in consequence (1) are sufficiently satisfactory.

By the way, it is confirmed that Fig.4 (a) and (c) differ very little from the true responses, respectively.

Next, the second stage identification based on these results will be shown.

Fig.5 shows the amplitude and the phase of constant-damping transforms for the finite approximate pulse transfer function $F_N(z)$ in the case of (Ex-1). The row axis is scaled in angular θ and its full scale is $(-2\pi, +2\pi)$. Fig.5 (a) and (b) are one of the case S/N ratio = 0.0 and S/N ratio = 0.1 with the damping term $r_0 = 1.0$, respectively. From these figures, the phase term θ_p can be read as $\pi/4$.

As an illustration for estimation of the damping term r_p, Fig.5 (c) and (d) are shown. From these figures, inversion of the phase at the origin, i.e., $\theta_p = 0$ can be observed, but on the neighbourhood $\pi/4$ no inversion of the phase can be seen.

From these facts and theorem 1, we can judge as $r_1 < p_1 = 0.980 < r_2$, where $r_1=0.978$ (Fig.5(d)) and $r_2=0.982$ (Fig.5(c)). That is to say, we can estimate the damping term r_p of p_1 as the mean value between one of the pre-inversion and one of the post-inversion.

Fig.6 shows the amplitude and the phase of constant-damping transforms for the ninth order system (Ex-2). Fig.6 (a) - (d) explain as like facts as Fig.5 (a) - (d).

Judging from the above discussions and figures, it is thought that fully satisfactory results have been obtained with only five times' or ten times' superpositions for considerably high order systems.

VI. Conclusion

In this paper, we have proposed an identification algorithm using FFT from first to last in the major operations to identify a linear discrete-time system with input/output data.

The proposed method is essentially one of identification methods in frequency domain because FFT algorithm is employed. Though generally, it has been thought that it is very difficult to identify the damping terms of the system poles or zeros, it has been shown that the damping terms are possible to be identified as well as the phase terms with the accuracy corresponding to frequency resolution by using the phase characteristics. Moreover, it has been shown that as a device so as to reduce the effects of the observation noise and the statistical fluctuation owing to finiteness of data, a kind of superposition in frequency domain using the correlation functions is effective and favourable in data processing.

It is thought that the proposed algorithm is applicable to the identification of the higher order systems because it has high speed and high accuracy in operations being inherent in FFT.

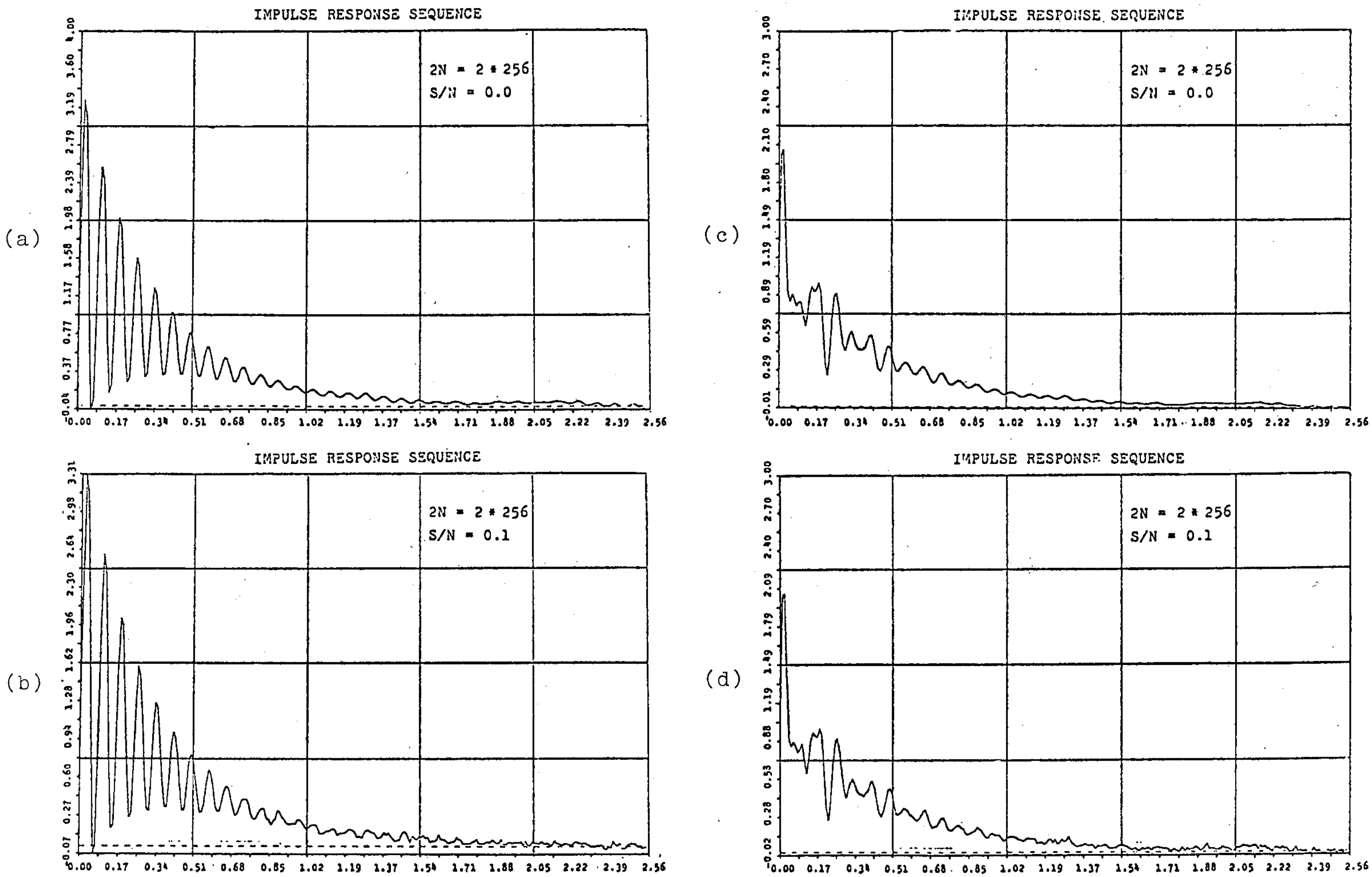

Fig.4. Impulse response sequences estimated in the first stage identification: (a), (b); for the third order system (Ex-1), (c),(d); for the ninth order system (Ex-2).

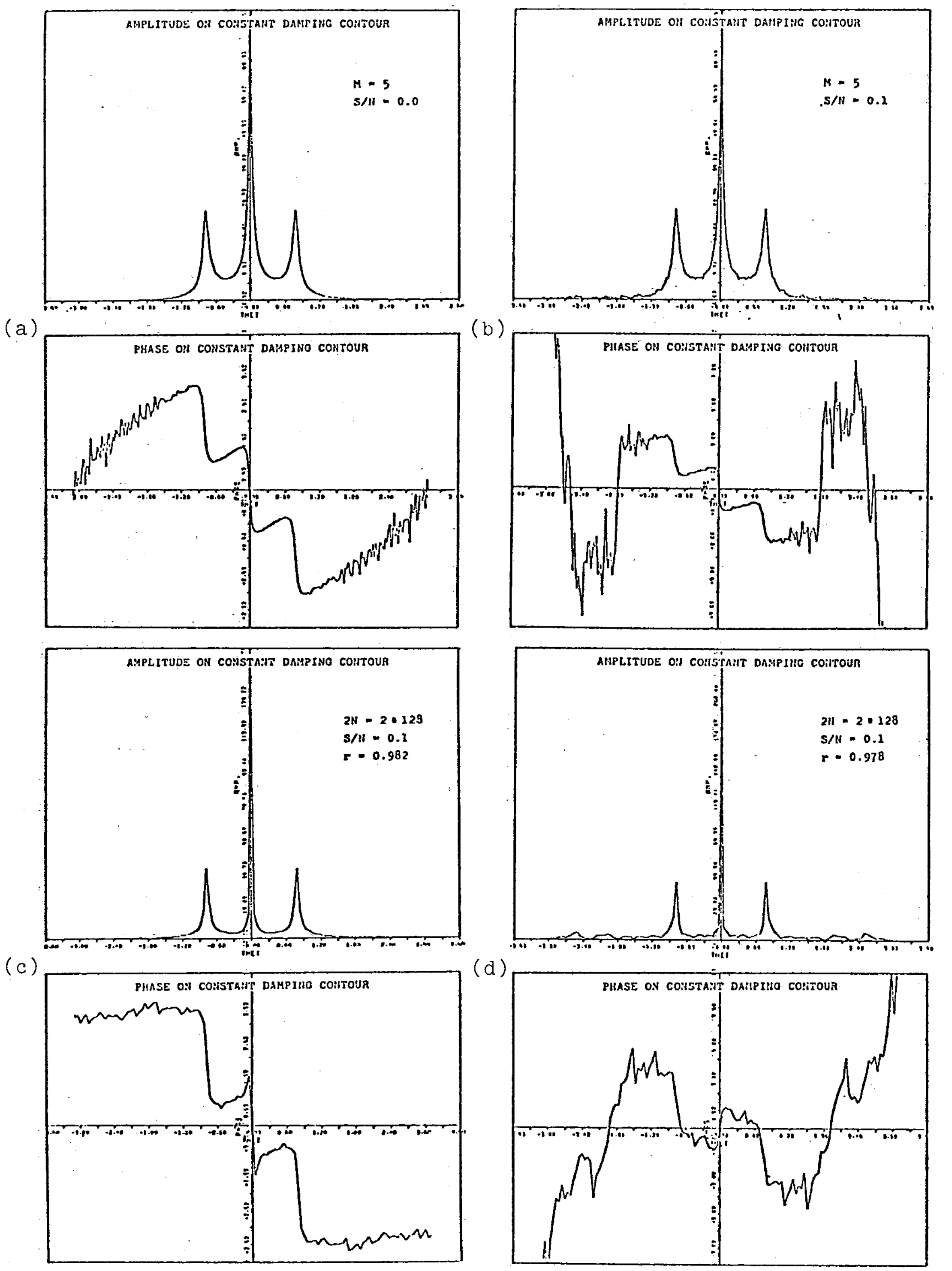

Fig.5. Amplitude spectra and phase spectra of constant-damping transforms for the third order system (Ex-1): (a),(b); with constant damping r = 1.0, (c),(d); close to a system's pole.

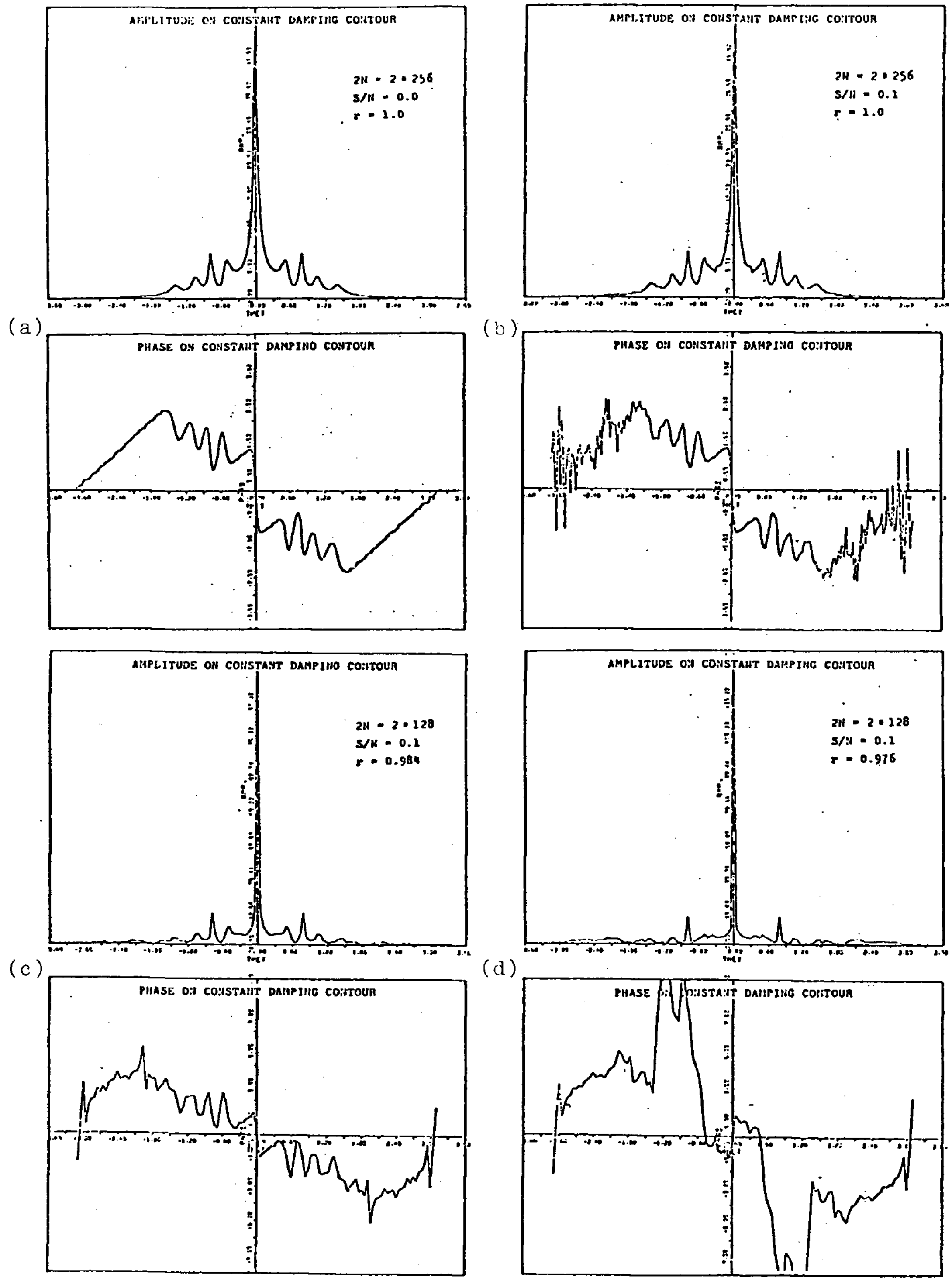

Fig.6. Amplitude spectra and phase spectra of constant-damping transforms for the ninth order system (Ex-2): (a),(b); with constant damping r = 1.0, (c),(d); close to a system's pole.

Acknowledgement

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The computer simulations have been carried out with Okayama University Computer Center Acos-700 System.

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(Appendix I) Proof of (13)

Considering $D_{2N}^{-1} = (1/2N)\cdot D_{2N}^{*}$, the transformation is thus

$$\begin{aligned}
\Gamma\cdot D_{2N}^{-1} &= \frac{1}{2N}\cdot\Gamma\cdot D_{2N}^{*} \\
&= \frac{1}{2N}\left\{({}_0u_{N0}^{T}D_{2N}^{*})^{T}, ({}_1u_{N0}^{T}D_{2N}^{*})^{T}, \ldots, ({}_{2N-1}u_{N0}^{T}D_{2N}^{*})^{T}\right\}^{T} \\
&= \begin{pmatrix} U_0^{*}(0), & U_0^{*}(1), & \ldots, & U_0^{*}(2N-1) \\ U_0^{*}(0)w^{-1}, & U_0^{*}(1)w^{-2}, & \ldots, & U_0^{*}(2N-1)w^{-(2N-1)} \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ U_0^{*}(0)w^{-(2N-1)}, & U_0^{*}(1)w^{-(2N-2)}, & \ldots, & U_0^{*}(2N-1)w^{-1} \end{pmatrix} \\
&= \frac{1}{2N}\cdot D_{2N}^{*}\cdot\Lambda(U_0^{*}) = D_{2N}^{-1}\cdot\Lambda(U_0^{*})
\end{aligned}$$

where $\Lambda(U_0^{*}) = \mathrm{diag}(U_0^{*}(0), U_0^{*}(1), \ldots, U_0^{*}(2N-1))$. Q.E.D.

(Appendix II) Proof of (32) in Theorem 1

Let us rewrite the assumption (31).

$$\varphi(z) = \arg\{\Sigma_1(z)\} + \arg\{\sum_{k=0}^{N-1} (p\cdot z^{-1})^k\} \tag{31}$$

In order to evaluate the change of $\varphi(z)$ in the vicinity of $z = p$, since the first term of (31) can be regarded as a constant, it is sufficient to consider only the second term.

Let this term be $\varphi_1(z)$. Then, $\varphi_1(z)$ is represented as (II-1).

$$\varphi_1(\theta) = \arg\left\{\frac{1 - q^N\cdot \exp(i\cdot N\cdot(\theta_p - \theta))}{1 - q\cdot\exp(i\cdot(\theta_p - \theta))}\right\} \tag{II-1}$$

where $z \triangleq r\cdot\exp(i\cdot\theta)$, $p \triangleq r_p\cdot \exp(i\cdot \theta_p)$, $q \triangleq r_p/r$. (II-2)

Well, since the phase difference $(\theta_p - \theta)$ is given by $k\cdot(2\pi/N)$ (k : integer) , $\exp(i\cdot N\cdot(\theta_p - \theta))$ equals to one. Therefore,

$$\varphi_1(\alpha) = \arg\left\{\frac{1 - q^N}{1 - q\cdot \exp(i\cdot\alpha)}\right\} = \tan^{-1}\left(\frac{q\cdot\sin\alpha}{1 - q\cdot\cos\alpha}\right) \tag{II-3}$$

where $\alpha \triangleq \theta_p - \theta$.

$$\frac{d\varphi_1(\alpha)}{d\alpha} = \left\{\frac{q\cdot(\cos\alpha - q)}{(1 - q\cdot\cos\alpha)^2}\right\}\cos^2\varphi_1(\alpha) \tag{II-5}$$

Putting $\alpha = 0$, (II-6) is obtained.

$$\left.\frac{d\varphi_1(\alpha)}{d\alpha}\right|_{\alpha=0} = \frac{q\cdot(1 - q)}{(1 - q)^2} = \frac{1}{1 - q} \tag{II-6}$$

Considering (II-4), (II-6) and the above discussion, (II-7) is obtained.

$$\left.\frac{d\varphi}{d\theta}\right|_{\theta=\theta_p} \cong \left.\frac{d\varphi_1}{d\theta}\right|_{\theta=\theta_p} = -\left.\frac{d\varphi_1}{d\alpha}\right|_{\alpha=0} \begin{array}{ll} < 0 & \text{for } q<1 \\ > 0 & \text{for } q>1 \end{array} \tag{II-7}$$

Q.E.D.

(Appendix III) Proof of (34) in Theorem 2

Considering the phase of (33) as like as (31), (33)' is obtained.

$$\varphi(z) = \arg\{\Sigma_2(z)\} + \arg\{\sum_{k=0}^{N-1} (\frac{1}{p_1-p_2})\cdot(p_1{}^{k+1} - p_2{}^{k+1})\cdot z^{-k}\} \tag{33'}$$

By the discussion similar to Appendix II , (III-1) is obtained.

$$\varphi_1(\theta) = \arg\left\{\left(\frac{1}{r_1 - r_2}\right)\cdot\sum_{j=1}^{2} \frac{(-1)^{j-1}\cdot r_j\ (1 - q_j{}^N)}{(1 - q_j\cdot\exp(\, i\cdot(\theta_p - \theta)))}\right\} \tag{III-1}$$

where

$$z \triangleq r\cdot\exp(\, i\cdot\theta)\ ,\ p_j \triangleq r_j\cdot\exp(\, i\theta_p),\ q_j \triangleq r_j/r\ ;\ j = 1,2. \tag{III-2}$$

By transforming (III-1) as well as in Appendix II, we have (III-3).

$$\varphi_1(\alpha) = \arg\left\{\frac{a_1 - a_2\ \exp(\, i\cdot\alpha\,)}{(1 - q_1\cdot\exp(\, i\cdot\alpha))(1 - q_2\cdot\exp(\, i\cdot\alpha))}\right\} \tag{III-3}$$

where $\alpha \triangleq \theta_p - \theta$,

$$a_1 \triangleq \{r_1\cdot(1-q_1{}^N) - r_2\cdot(1-q_2{}^N)\}/(r_1-r_2)\quad ,$$

$$a_2 \triangleq \{r_1\cdot q_2\cdot(1-q_1{}^N) - r_2\cdot q_1\cdot(1-q_2{}^N)\}/(r_1-r_2)\ . \tag{III-4}$$

Moreover, letting $q_0 \triangleq a_2/a_1$, $s_0 \triangleq -1$, s_1, $s_2 \triangleq 1$, we have (III-5).

$$\varphi_1(\alpha) = \sum_{j=0}^{2} s_j\cdot\tan^{-1}\left(\frac{q_j\ \sin\alpha}{1 - q_j\ \cos\alpha}\right) \tag{III-5}$$

Let the first, second and last term of (III-5) be $\psi_1(\alpha)$, $\psi_2(\alpha)$ and $\psi_3(\alpha)$, respectively. Then, in the like manner of induction of (II-5) in Appendix II, we can obtain (III-6).

$$\frac{d\,\varphi_1(\alpha)}{d\ \alpha} = \sum_{j=0}^{2} s_j\left\{\frac{q_j\cdot(\cos\alpha - q_j)}{(1 - q_j\cdot\cos\alpha)}\right\}\cos^2\psi_j(\alpha) \tag{III-6}$$

Putting $\alpha = 0$, (III-7) is obtained.

$$\left.\frac{d\,\varphi_1(\alpha)}{d\ \alpha}\right|_{\alpha=0} = \sum_{j=0}^{2} s_j\cdot q_j/(1 - q_j) \tag{III-7}$$

Now, assuming that q_1 and q_2 nearly equal to one, respectively, the following approximate equations can be utilized.

$$q_j{}^N = \{1 - (1 - q_j)\}^N \cong 1 - N\cdot(1 - q_j)\quad ;\ j = 1,2 \tag{III-8}$$

Arranging (III-7) with (III-4) and (III-8), we have (III-9).

$$\left.\frac{d\,\varphi_1(\alpha)}{d\ \alpha}\right|_{\alpha=0} = \frac{1}{(1 - q_1)(1 - q_2)} - 1 \tag{III-9}$$

In the similar manner to Appendix II, within which (III-8) is allowed, the following relations can be obtained.

$$\begin{aligned}\left.\frac{d\,\varphi(\theta)}{d\ \theta}\right|_{\theta=\theta_p} \cong \left.\frac{d\,\varphi_1(\theta)}{d\ \theta}\right|_{\theta=\theta_p} & \quad < 0 \quad \text{for } (1 > q_1 > q_2) \\ = -\left.\frac{d\,\varphi_1(\alpha)}{d\ \alpha}\right|_{\alpha=0} & \quad > 0 \quad \text{for } (q_1 > 1 > q_2) \\ & \quad < 0 \quad \text{for } (q_1 > q_2 > 1)\end{aligned} \tag{III-10}$$

Q.E.D.