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A Simulator of Waiting Line Problems

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Demand and supply of service are complicatedly related with the balance between fixed capital and circulating capital, movement to the left or right side from a break-even point, and other factors. If there is the disproportion between demand and supply of service, the waiting line to take a service will vary, and in some cases, fixed equipment will not be employed effectively.

This report presents trial manufacture of the experimental equipment for waiting line problems.

§ 1. Introduction

We make the following assumptions in the experiement of waiting line problem.

Assumption 1

The stream of arrival of units* show the simplest stream which possess a single parameter. The unit is served in order of arrival at a channel. The length of service is always independent as a rule.

The arrival of units applies to **Poisson Distribution** which takes a optional length **k/n** (**k**=1, 2, —) in optional time interval (0, 1), and shows the unit number in the interval of length "t" with a parameter "λt" as follows

$$U_k(t)=\mathbf{exp}(-\lambda t)\cdot(\lambda t)^k/\mathbf{k}!,\ [\mathbf{k}=0, 1, 2\cdots] \tag{1}$$

The length of service time "l" applies to an Exponential Distribution as follows

$$\mathbf{p}(l<t)=\mathbf{exp}(-\mu t),\ (t<0,\ \mu>0) \tag{2}$$

Assumption 2

The mechanism of service station can be classified into two groups.

Group 1: If the channel is unoccupied when a unit arrives at this channel, the unit will occupy the channel and receive a service there.

If the channel is occupied when a unit arrives there, the unit will be refused, and process of system after receiving a service will be brought back to a starting point.

Group 2: Another mechanism preserves (as a next demandant of service) a unit which arrives at the channel when it is occupied, and when unoccupied, at first gives unit the channel to a unit preferentially.

* The unitmeans an article, a manufacture, a man, a customer and the others which arrive to service channel.

These two mechanisms are a little different in existing probability.

In the former, the formation of waiting line is not recognized, and in the latter the length of waiting line has a close connection with the acquisition of service.

In this problem, we consider a case where there is the waiting line of some length. If the length of waiting line reaches "m", so far as the system is not unchanged. (for instance, a unit among the "m" gets through the service), next unit will be refused to take a service and the unit progresses as if it does not reach at the beginning.

Thus, assumption 2 makes a waiting line within the limits that the length of waiting line is less than "m" and "k" pieces of unit occupy the space in the waiting line at the optional time point, and the process of probability variable "$\mathbf{N}(\mathbf{t})$" to possess the space at the optional time point "t" follows the probability process.

"n" pieces of the existent probability "$\mathbf{P}_n$", in the system of such cases where assumption 1 and 2 are included, is given as equations (3), (4).

$$\mathbf{P}_n=\rho^n\mathbf{P}_0,\ (0\leq n\leq m) \tag{3}$$

$$\mathbf{P}_0=(1-\rho)/(1-\rho^{m+1}) \tag{4}$$

In the same way, extinction rates on the maximum length of waiting line "m" are given as follows

$$\mathbf{P}_m=\rho^m\mathbf{P}_0 \tag{5}$$

§ 2. Experimental Equipment

The simplest form of model which can be used has the maximum length of waiting line m=2. The flow-chart of this system is shown in Figure 1, and Figure 2 represents the circuit in which the relays are used.

Experimenters can set up and operate at their option of the arrival times and distribution of service time of a unit.

The mean service time and the time required for an experiment correlate through the form of distribution, and the simple repetitional expęriments is apt to cause the operational errors. For instance, we have found 2~3 per cent operational error during 150 minutes of experiment.

On the contrary, the influences of the sense for experimenter to set up the times and of precision of timer for service times increase if the experimenter decreases the mean service time in order to shorten the experimental time. For instance, the relations among the parameter "μ" (mean service rate), the experimental time, and traffic density "$\rho=\lambda/\mu$" are shown in Table 1, and in the case of the "$\mu=6$", 95 per cent "**P** ($l<$t)$=0.95$" of the service times distribution becomes more than 0.5 seconds, and it is beyond 30 seconds at "**P** ($l>$t)$=0.05$". In particular, the fact that must be set up the service times below the 0.5 seconds for 5 per cent of all trial units mumber, means that the timers will be of no use in this case, for the simple timers on the market show a gross error below the 2 seconds in set up times.

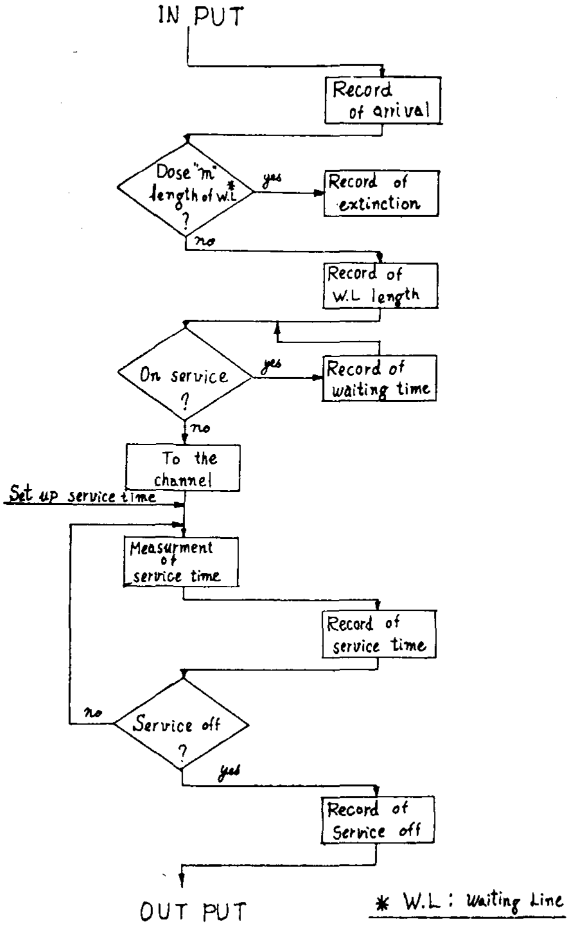

Fig. 1 Flow-chart of the system.

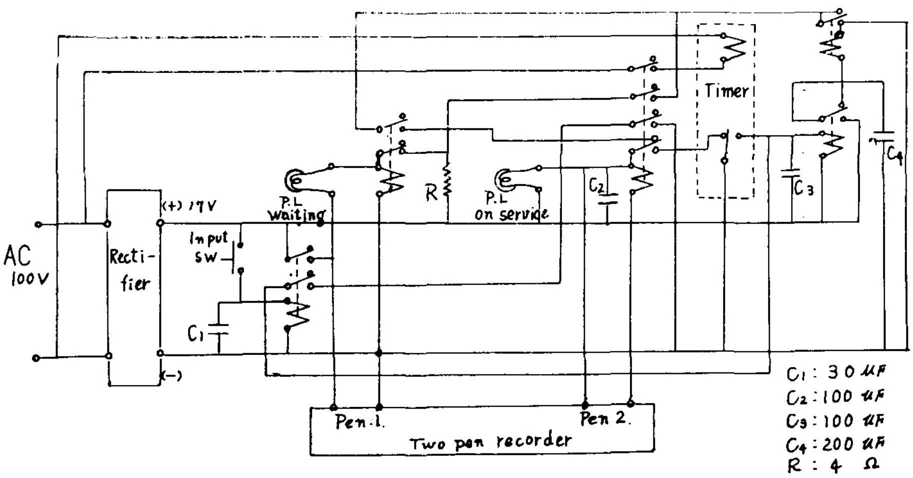

Fig. 2 Circuit diagram of experimental equipment.

Therefore, the experimental model uses on "$\mu\leq 2$ ($\mu>0$)" as the Table 1.

As mentioned above, on the experiment for stationary of thę waiting line, if the trial number of times per experiment is one hundred times, the time requires about two hours at "$\rho=0.4$", and about four hours at the "$\rho=0.2$", The situations of a system under the experiment are made record by the recorder from the beginінig to the end. The results of experiments are recorded in the chart as the variation of potential in Figure 3.

§ 3. Result of Experiment

The distribution of service times was represented as an exponential distribution of parameter "$\mu=2$", and the distribution of arrival time was represented with the various values of paremeter "λ" by using traffic density "$\rho=0.2$, 0.4, 0.6, 0.8, 1.0" and each distribution is determined by **Monte Carlo Methods.** As the result, the arrival time and service time were obtained in the table of appendix.

ρ \ μ	2 (min)	4	6	8	10
0.2 ⋮ 1.0	250 ⋮ 50	125 ⋮ 25	$83\frac{1}{3}$ ⋮ $16\frac{1}{3}$	$62\frac{1}{2}$ ⋮ $12\frac{1}{2}$	50 ⋮ 10
τ_s	1.5 sec	0.75 sec	0.5 sec	0.48 sec	0.3 sec
τ_l	90 sec	45 sec	30 sec	22.5 sec	18 sec

Table 1. Experimental times for each "p" on arrival number of trial =100, and caluculative values of service time.

Explanation: Calculative values of service time = arrival number of trial $1/(\rho\cdot\mu)$.

π_s: Maximum time when arrival unit of 5 per cent receives shot service=in **exp** $(-\mu t)=0.95$, "t" 60 sec.

π_l: Minimum time when arrival unit of 5 per cent receives long service =**exp** $(-\mu t)=0.05$, "t" 60 sec.

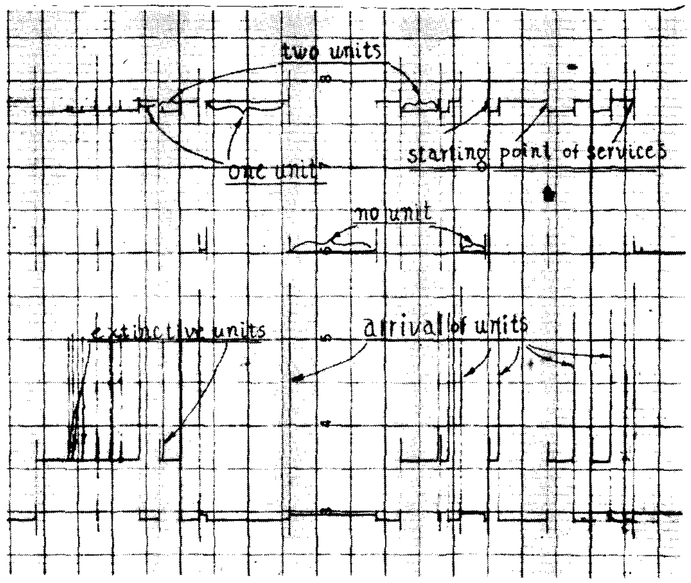

Fig. 3 Result of experiment.

Thus, arranging the result of experiments as to each traffic density, the results are summarized in Table 2.

The table 2 (a) (b) (c) is divided into ten periods, and the each experimental time as to each traffic density "ρ_i", and each rows represent the period, each probabilities are represent as follows.

Existential probability of no unit = $\mathbf{P}'_0$ → Table (a)

Exsitential probability of one unit = $\mathbf{P}'_1$ → Table (b)

Existential probability of two unit = $\mathbf{P}'_2$ → Table (c)

The table (d) shown extinction rates $\mathbf{P}'_m$ of arrival units, and the last row indicates the total average probability on each traffic density.

The other hand, theoretical values are evaluated from formula (3), (4), (5). And comparing these values with each total average of evry probabilities on the basis of experiment, it is shown as Figure 4.

Generally, theoretical probability ($\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_m$) approaches to 1/3 as traffic density closes to 1. But there is some difference between the theoretical value and experimental value as to processes of approach.

§ 4. TestingStatistical Hypothesis of Experimental values

The experimental value of table 2 shows the result based on arrival time distributions and a service time distributions which we obtained by using **Monte Carlo Methods.** Therefore, they can be considered as a part of samples in experimental results of infinite times. Consequently, on each traffic densities "ρ_i", the total average "$\bar{x}_{P_m\cdot\rho_i}$", the variable number "$V_{P_m\cdot q_i}$" from period No. 1 to No. 10 of the probability "$\mathbf{P}_n\cdot\rho$" with "n" units in the system at a time

Table 2 Experimental values of probabilities.

(a) P_0' = Time of existential of no unit / Time of a period

P_i \ Period No	1	2	3	4	5	6	7	8	9	10	Total average	$\frac{1-P_i}{1-P_i^{3+1}}$
1.0	0.29	0.38	0.34	0.38	0.21	0.34	0.75	0.32	0.14	0.42	0.36	0.333
0.8	0.33	0.65	0.31	0.29	0.78	0.69	0.66	0.14	0.42	0.69	0.45	0.410
0.6	0.35	0.81	0.45	0.59	0.61	0.60	0.42	0.37	0.45	0.40	0.51	0.510
0.4	0.63	0.74	0.48	0.72	0.86	0.88	0.52	0.74	0.80	0.36	0.67	0.641
0.2	0.83	0.89	0.66	0.85	0.96	0.90	0.69	0.86	0.73	0.56	0.79	0.808

(b) P_1' = Time of existential of one unit / Time of period

P_i \ Period No	1	2	3	4	5	6	7	8	9	10	Total average	$\frac{(1-P_i)P_i}{1-P_i^{3+1}}$
1.0	0.22	0.46	0.26	0.26	0.50	0.34	0.23	0.35	0.33	0.39	0.33	0.333
0.8	0.37	0.35	0.39	0.33	0.22	0.29	0.31	0.58	0.42	0.21	0.35	0.328
0.6	0.39	0.15	0.22	0.18	0.29	0.34	0.30	0.52	0.54	0.33	0.33	0.306
0.4	0.24	0.24	0.25	0.18	0.10	0.12	0.36	0.21	0.18	0.34	0.22	0.256
0.2	0.17	0.11	0.30	0.11	0.04	0.10	0.29	0.12	0.21	0.35	0.18	0.161

(c) P_2' = Time of existential of two unit / Time of period

P_i \ Period No	1	2	3	4	5	6	7	8	9	10	Total average	$\frac{(1-P_i)P_i^2}{1-P_i^{3+1}}$
1.0	0.48	0.15	0.41	0.36	0.29	0.32	0.02	0.34	0.53	0.21	0.31	0.333
0.8	0.36	0.00	0.30	0.38	0.50	0.02	0.03	0.27	0.16	0.09	0.21	0.262
0.6	0.26	0.04	0.32	0.23	0.10	0.05	0.23	0.11	0.01	0.27	0.18	0.184
0.4	0.13	0.02	0.28	0.10	0.05	0.00	0.00	0.05	0.04	0.31	0.11	0.103
0.2	0.00	0.00	0.03	0.04	0.00	0.00	0.01	0.02	0.08	0.09	0.03	0.032

(d) (Extinctive rate): P_m' = Extinctive number / Arrival number

P_i \ Period No.	1	2	3	4	5	6	7	8	9	10	Total average	$\frac{(1-P_i)P_i^3}{1-P_i^{3+1}}$
1.0	0.44	0.23	0.27	0.20	0.11	0.17	0.00	0.42	0.30	0.33	0.25	0.333
0.8	0.40	0.00	0.27	0.33	0.40	0.00	0.00	0.27	0.00	0.00	0.19	0.262
0.6	0.17	0.00	0.31	0.14	0.13	0.00	0.00	0.07	0.00	0.14	0.10	0.184
0.4	0.11	0.00	0.07	0.22	0.11	0.00	0.00	0.10	0.17	0.23	0.10	0.103
0.2	0.00	0.00	0.00	0.08	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.032

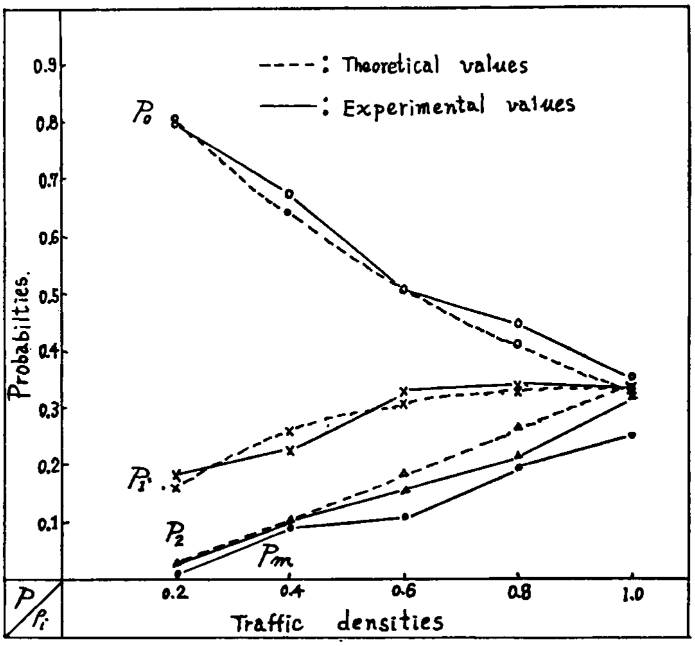

Fig. 4 Difference between theoretical and experimental probabilities.

Table 3. Values of "t".

P_n' \ P_i	0.2	0.4	0.6	0.8	1.0
P_0'	0.113	0.203	0.037	0.184	0.177
P_1'	0.204	0.447	0.194	0.188	0.044
P_2'	0.241	0.047	0.208	0.352	0.151
P_m	1.000	0.007	0.907	0.627	0.667

point is as follows

$$\bar{x}_{P_m\cdot P_i} = 1/n \times \sum_n x_{P_n\cdot P_i} \tag{6}$$

$$V_{P_n\cdot P_i} = \sum (x_{P_n\cdot P_i})^2 (-\sum x_{P_\rho\cdot P_i})^2/n \tag{7}$$

and Fisher's "t" is given

$$t_{P_n\cdot P_i} = [\bar{x}_{P_n\cdot P_i} - \rho^n(1-\rho_i)/(1-\rho^{m+1}] / \sqrt{v/n} \tag{8}$$

Caluculating the "$t_{P_n\cdot P_i}$", they are shown at Thble 3, and on 95 per cent of the level of significance, they are as follows

$$t_{P_n\cdot P_i} < t(9, 0.05) = 2.262.$$

Hence, the experimental equipment and the data that were obtained by the experiment have significance, and they can be trusted as the result of experiment on the theory of waiting line.

Table of Appendix (1)

Table of Poisson arrival distribution.

p=0·2

No.	Arrival time	No.	Arrival time	No.	Arrival time	No.	Arrival time
1	0sec	26	4262	51	7226	76	11073
2	123	27	4304	52	7457	77	11121
3	438	28	4333	53	7585	78	11732
4	508	29	4381	54	7830	79	11852
5	680	30	4387	55	8050	80	11909
6	1050	31	4419	56	8112	81	12365
7	1541	32	4458	57	8789	82	12405
8	1594	33	4512	58	8852	83	12723
9	2013	34	4620	59	8958	84	12777
10	2066	35	4624	60	9136	85	12798
11	2281	36	4685	61	9750	86	13104
12	2319	37	4739	62	9792	87	13291
13	2335	38	4806	63	9932	88	13456
14	2414	39	4854	64	10111	89	13502
15	2441	40	5053	65	10141	90	13550
16	2719	41	5347	66	10170	91	13582
17	2947	42	5604	67	10187	92	13693
18	3010	43	5666	68	10204	93	13987
19	3146	44	5847	69	10211	94	14056
20	3192	45	6006	70	10276	95	14102
21	3318	46	6044	71	10351	96	14136
22	3495	47	6183	72	10651	97	14534
23	3612	48	6204	73	10869	98	14655
24	3925	49	6686	74	10949	99	14711
25	4092	50	7094	75	11003	100	18454

(Experimental time = 250 min)

p = 0.4

No.	Arrival time	No.	Arrival time	No	Arrival time	No	Arrival time
1	0sec	26	1722	51	3367	76	5419
2	7	27	1723	52	3550	77	5472
3	21	28	1757	53	3656	78	5577
4	43	29	1786	54	3686	79	5644
5	208	30	1938	55	3792	80	5705
6	353	31	1970	56	3957	81	5805
7	442	32	1989	57	3978	82	6006
8	517	33	2031	58	3994	83	6185
9	526	34	2198	59	4057	84	6229
10	614	35	2201	60	4095	85	6230
11	757	36	2226	61	4314	86	6233
12	771	37	2362	62	4452	87	6396
13	790	38	2396	63	4548	88	6542
14	831	39	2422	64	4597	89	6777
15	959	40	2521	65	4654	90	6837
16	1048	41	2674	66	4705	91	6839
17	1169	42	2706	67	4885	92	6850
18	1290	43	2750	68	4891	93	6892
19	1335	44	2758	69	4945	94	7052
20	1408	45	2813	70	4992	95	7104
21	1479	46	3055	71	4206	96	7126
22	1488	47	3059	72	5277	97	7133
23	1578	48	3007	73	5282	98	7175
24	1615	49	3216	74	5295	99	7227
25	1713	50	3230	75	5401	100	7348

(Experimental time = 125 min)

Table of appendix (2)

Table of Poisson arrival distribution.

p = 0.6

No.	Arrival time	No.	Arrival time	No.	Arrival time	No	Arrival time
1	0 sec	26	1394	51	2647	76	3622
2	10	27	1417	52	2708	77	3645
3	48	28	1421	53	2714	78	3669
4	113	29	1432	54	2753	79	3725
5	146	30	1456	55	2756	80	3740
6	240	31	1463	56	2759	81	3872
7	257	32	1500	57	2761	82	3929
8	274	33	1530	58	2839	83	3941
9	303	34	1669	59	2862	84	3957
10	317	35	1768	60	2913	85	3967
11	455	36	1774	61	2918	86	3988
12	483	37	1888	62	2937	87	4064
13	639	38	1892	63	2956	88	4111
14	680	39	1929	64	3023	89	4120
15	707	40	2094	65	3078	90	4261
16	719	41	2254	66	3143	91	4345
17	719	42	2361	67	3258	92	4418
18	846	43	2370	68	3265	93	4438
19	967	44	2426	69	3330	94	4598
20	1058	45	2476	70	3424	95	4656
21	1071	46	2489	71	3455	96	4736
22	1101	47	2509	72	3485	97	4745
23	1158	48	2540	73	3563	98	4773
24	1181	49	2548	74	3575	99	4872
25	1376	50	2575	75	3583	100	4999

(Experimental time ≒ 83 min)

p = 0.8

No.	Arrival time	No.	Arrival time	No	Arrival time	No.	Arrival time
1	0 sec	26	914	51	1653	76	2644
2	48	27	935	52	1677	77	2660
3	65	28	940	53	1683	78	2802
4	144	29	943	54	1690	79	2819
5	156	30	983	55	1806	80	2826
6	201	31	1033	56	1813	81	2856
7	229	32	1050	57	1837	82	2860
8	237	33	1099	58	1898	83	2975
9	242	34	1141	59	1944	84	3090
10	384	35	1144	60	1948	85	3091
11	439	36	1180	61	2080	86	3132
12	451	37	1193	62	2101	87	3160
13	535	38	1213	63	2121	88	3170
14	568	39	1235	64	2184	89	3178
15	611	40	1250	65	2198	90	3283
16	687	41	1339	66	2277	91	3289
17	713	42	1380	67	2367	92	3304
18	738	43	1406	68	2369	93	3432
19	768	44	1427	69	2372	94	3464
20	773	45	1454	70	2386	95	3468
21	784	46	1483	71	2399	96	3575
22	829	47	1489	72	2453	97	3605
23	877	48	1493	73	2460	98	3614
24	888	49	1517	74	2531	99	3683
25	895	50	1688	75	2586	100	3708

(Exporimental time ≒ 62.5 min)

Table of appendix (3)

Table of Exponential service distribution.

p = 1.0

No.	Arrival time	No.	Arrival time	No.	Arrival time	No.	Arrival time
1	0sec	26	694	51	1538	76	2298
2	7	27	702	52	1566	77	2315
3	22	28	703	53	1586	78	2321
4	37	29	725	54	1588	79	2336
5	69	30	744	55	1679	80	2349
6	135	31	757	56	1687	81	2381
7	166	32	857	57	1725	82	2414
8	179	33	870	58	1762	83	2420
9	188	34	907	59	1777	84	2446
10	318	35	1030	60	1783	85	2458
11	397	36	1066	61	1851	86	2472
12	413	37	1074	62	1858	87	2518
13	429	38	1094	63	1898	88	2527
14	450	39	1156	64	1914	89	2551
15	467	40	1218	65	1982	90	2594
16	476	41	1233	66	2004	91	2637
17	488	42	1257	67	2048	92	2732
18	500	43	1275	68	2071	93	2747
19	509	44	1333	69	2077	94	2749
20	511	45	1368	70	2105	95	2762
21	538	46	1422	71	2169	96	2808
22	573	47	1440	72	2200	97	2948
23	599	48	1473	73	2234	98	2970
24	602	49	1509	74	2287	99	2986
25	664	50	1518	75	2288	100	2991

(Experimental time = 50 min)

u = 2

No.	Service time	No.	Service time	No.	Service time	No.	Service time
1	78	26	63	51	12	76	30
2	6	27	12	52	9	77	12
3	30	28	24	53	6	78	12
4	84	29	9	54	39	79	90
5	12	30	15	55	6	80	24
6	48	31	54	56	3	81	27
7	3	32	18	57	36	82	12
8	36	33	60	58	51	83	15
9	3	34	18	59	48	84	105
10	27	35	33	60	42	85	24
11	9	36	3	61	33	86	54
12	18	37	75	62	69	87	57
13	6	38	3	63	51	88	17
14	21	39	9	64	24	89	45
15	42	40	21	65	39	90	5
16	3	41	3	66	3	91	5
17	21	42	27	67	18	92	6
18	15	43	6	68	18	93	30
19	66	44	21	69	15	94	72
20	12	45	3	70	21	95	33
21	45	46	3	71	57	96	8
22	12	47	5	72	9	97	9
23	6	48	9	73	6	98	18
24	30	49	3	74	9	99	8
25	96	50	9	75	5	100	27