Flow of Rarefied Vapour past a Liquid Sphere

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SYNOPSIS

This paper deals with the low Mach numbers flow of a rarefied vapour past a liquid sphere accompanied with condensation and evaporation at its surface. The linearized Bhatnager-Gross-Krook(B-G-K) equation is used for the analysis, and from it the integral equations of the density, temperature and flow velocities are derived. These integral equations are solved numerically over a wide range of the Knudsen number covering from the slip flow to the nearly free molecular flow. The drag on the sphere is also calculated and is compared with that of previous work.

1. INTRODUCTION

The motion of or the flow around a very small particle suspended in a gas is an interesting problem in aerosol science and technology. A uniform flow past a solid spherical particle in a rarefied gas is a basic problem to understand the motion of aerosols, and this problem has been extensively studied.[1∿5] The drag on the sphere was calculated by a variational method for the whole range of the Knudsen number,[1] by the Knudsen iteration method

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for large Knudsen numbers,[2,3] and by the asymptotic theory for small Knudsen numbers.[4] Lea and Loyalka[5] calculated not only the drag on the sphere but also the flow field over a wide range of the Knudsen number.

When the particle is not solid but liquid, the condensation or evaporation usually takes place at its surface. This phenomenon will inevitably effect the velocity or temperature field over the liquid surface and hence the drag, too. The drag on a volatile liquid sphere was calculated for small Knudsen numbers[6∿8] and is found to be smaller than that of the solid sphere. The drag for a free molecular flow was obtained by Brock.[9] A variational method was applied to get the drag on a liquid sphere for a wide range of the Knudsen number.[10∿12] The variational method is very useful and rather simple to calculate the total quantities such as the drag, but it dose not provide the results for the local quantities such as the velocity . It is interesting and important for the investigation of the motion of aerosol to obtain the local quantities and to know how the rarefaction affects the temperature or velocity field.

In the present paper, we shall consider the low speed flow of a rarefied vapour past a liquid sphere, on which the condensation or evaporation takes place. The method of analysis is the same as is developed in previous papers.[13∿15] That is, we derive the simultaneous integral equations for the density, flow velocities, and temperature from the linearized B-G-K equation[16] by matching with the Stokes solution which is valid far from the body. These integral equations are solved numerically for a wide range of the Knudsen number covering from the slip flow to the nearly free molecular flow. We give the results for the drag as well as the velocity, density and temperature distributions.

2. FUNDAMENTAL EQUATIONS

We take a spherical droplet of radius a in its rarefied vapour. Let the number density, velocity, temperature, and pressure in the free stream be n_0, Q_0, T_0, and p_0, respectively. The temperature of the sphere is taken to be a constant T_w. We consider the steady problem.

It is assumed that the uniform speed Q_0 is small compared with the most probable speed $C_m = \sqrt{2kT_0/m}$, where k is the Boltzmann constant, and m the mass of a molecule. We also assume that the differences of the temperature between T_0 and T_w, and of the pressure between p_0 and p_w, which is the saturation pressure at T_w, are small. Then, we can linearize both the fundamental equations and the boundary conditions. We here employ the B-G-K equation. The linearized version of this equation is written in the following forms:

$$K\underline{v}\cdot\frac{\partial\phi}{\partial\underline{r}} = \phi_e - \phi, \tag{1}$$

$$\phi_e = \sigma + 2\underline{v}\cdot\underline{q} + \omega(\underline{v}^2 - \frac{3}{2}), \tag{2}$$

$$[\sigma,\ \underline{q},\ \frac{3}{2}(\sigma+\omega)=\frac{3}{2}\xi] = \int[1,\ \underline{v},\ \underline{v}^2]\phi E d\underline{v}, \tag{3}$$

$$E=\pi^{-3/2}\exp(-\underline{v}^2), \tag{4}$$

where $n_0C_m^{-3}E(1+\phi)$ is the distribution function of the molecular velocity, C_m the most probable speed, $n_0C_m^{-3}E(1+\phi_e)$ the local Maxwellian distribution function, $C_m\underline{v}$ the molecular velocity, $a\underline{r}$ the position vector, $n_0(1+\sigma)$ the number density, $T_0(1+\omega)$ the temperature, $p_0(1+\xi)$ the pressure, $C_m\underline{q}$ the flow velocity, $K=\sqrt{\pi}l/(2a)$ the Knudsen number, and l the mean free path.

As for the condition of the liquid droplet, it is assumed that the droplet has a constant radius a and a constant temperature T_w. We also assume that the molecules leaving the surface of the sphere have the Maxwellian distribution with temperature T_w and number density $n_w=p_w/(kT_w)$. Then, the distribution function for the reflected molecules at the surface is given by

$$\phi_{\underline{v}\cdot\underline{n}>0} = \xi_w + (\underline{v}^2 - \frac{5}{2})\omega_w \qquad (r=1), \tag{5}$$

where $\omega_w=(T_w-T_0)/T_w$, $\xi_w=\sigma_w+\omega_w=(p_w-p_0)/p_0$, and $\underline{n}$ the unit normal to the surface.

The uniform condition at infinity is given by

$$(q_r,\ q_\theta,\ q_\psi) \to S(\cos\theta,\ \sin\theta,\ 0),\quad (\sigma,\ \omega) \to (0,\ 0)\quad (\text{as } r\to\infty), \tag{6}$$

in the spherical polar coordinates (r,θ,ψ), and $S=Q_0/C_m$ is the speed ratio.

3. ANALYSIS

The method of analysis is the same as is developed in the previous work.[13∿15] In the field far from the sphere compared with the mean free path, the distribution function is close to an equilibrium because of a lot of intermolecular collisions. Therefore, we can treat the Stokes equations for the mean quantities instead of Eq.(1) for the distribution function. It is easy to get a proper solution of the Stokes equation which satisfies the condition at infinity (6) and also provides the form suggested by the boundary condition (5) at the surface. The results are given by

$$\sigma_S = -\frac{K(A+C)}{r^2}S\cos\theta - \frac{B_1\xi_w + B_2\omega_w}{r}, \tag{7}$$

$$q_{r,S} = \{1 - \frac{A}{r} + \frac{B}{r^3}\}S\cos\theta + \frac{A_1\xi_w + A_2\omega_w}{r^2}, \tag{8}$$

$$q_{\theta,S} = -\{1 - \frac{A}{2r} - \frac{B}{2r^3}\}S\sin\theta, \tag{9}$$

$$\omega_S = \frac{KC}{r^2}S\cos\theta + \frac{B_1\xi_w + B_2\omega_w}{r}, \tag{10}$$

where A, B, C, A_i and B_i $(i=1,2)$ are unknown constants to be determined by matching the Stokes flow with the flow near the sphere.

In the region adjacent to the sphere, whose extent is of the order of the mean free path, we must treat Eq.(1). We call this the kinetic region. Integrating Eq.(1) formally along a characteristic, we have

$$\phi = \phi_w \exp\{-\frac{1}{Kv}(t-t_w)\} + \frac{1}{Kv}\int_{t_w}^{t}\phi_e \exp\{-\frac{1}{Kv}(t-t')\}\,dt', \tag{11}$$

where the subscript w denotes the quantities at the boundary, and $v=|\underline{v}|$. The value ϕ_w is given by Eq.(5) if t_w is on the surface of the sphere, whereas it is taken to be zero when $t_w=\infty$. Substitution of Eq.(11) into Eq.(3) leads to the simultaneous integral equations for σ, $\underline{q}$ and ω. Considering that σ, $\underline{q}$ and ω approach the Stokes solution given by Eqs.(7)∿(10) outside the kinetic region, we may assume the solutions in the kinetic region in the following forms:

$$\sigma = \sigma_S + \Sigma(r)S\cos\theta + \xi_w\Sigma_1(r) + \omega_w\Sigma_2(r), \tag{12}$$

$$q_r = q_{r,S} + KQ_r(r)S\cos\theta, \tag{13}$$

$$q_\theta = q_{\theta,S} - KQ_\theta(r)S\sin\theta, \tag{14}$$

$$\omega = \omega_S + K\Theta(r)S\cos\theta + \xi_w\Theta_1 + \omega_w\Theta_2(r), \tag{15}$$

where $\Sigma(r)$, $Q_r(r)$, $Q_\theta(r)$, $\Theta(r)$, $\Sigma_i(r)$ and $\Theta_i(r)$ $(i=1,2)$ are the correction functions in the kinetic region and hence should vanish outside this region. We put these forms (12)∿(15) into the integral equations for σ, ω and $\underline{q}$. Since the derived equations are linear, the solutions can be superimposed. That is, the flow field caused by the uniform flow S is separated from the thermal field due to the temperature difference ω_w and ξ_w. The solution depending on ω_w and ξ_w has been obtained already[15] and hence we here only solve the equations for Σ, $\underline{Q}$ and Θ. These have the following forms:

$$\sqrt{\pi}G_i(r) = G_{K,i}(r) + \frac{2}{K}\left\{\int_0^\beta d\chi\int_{r-1}^{t_0}dt + \int_\beta^\pi d\chi\int_{r-1}^{r+1}dt - \int_0^\beta d\chi\int_{r+1}^{\infty}dt\right\}F_{A,i}$$
$$+ 2\int_1^\infty ds\int_0^{\psi_0} F_{K,i}\left(\frac{s}{t}\right)H_{K,i}d\psi \quad (i=1\sim4), \tag{16}$$

$$(G_1,\ G_2,\ G_3,\ G_4) = K(\Sigma,\ Q_r,\ -Q_\theta,\ \tfrac{3}{2}\Theta), \tag{17}$$

$$G_{K,1} = -\frac{4K}{r^2}[CJ_4(\eta) - (A + \tfrac{5}{2}C)J_2(\eta)], \tag{18}$$

$$G_{K,2} = \frac{8}{3r^3}\Big\{- K^2[CJ_6(\eta) - (A + \tfrac{5}{2}C)J_4(\eta) + \tfrac{2}{5}AJ_6(\eta)]$$
$$- K(r-1)[CJ_5(\eta) - (A + \tfrac{5}{2}C)J_3(\eta) + \tfrac{2}{5}AJ_5(\eta)]$$
$$- (r-1)^2\tfrac{A}{5}J_4(\eta)\Big\} - \tfrac{8}{3}(1 - \frac{A}{r} + \frac{B}{r^3})J_4(\eta)$$
$$+\frac{4}{3K}\int_{r+1}^{\infty}\Big\{\frac{1}{K\tau^2}[(A + \tfrac{5}{2}C)J_2(\tau) - CJ_4(\tau)]$$
$$+ 2J_3(\tau)[1 - (\frac{1}{t} - \frac{r^2}{5t^3})A + \frac{B}{t^3}]\Big\}dt, \qquad (19)$$

$$G_{K,3} = \frac{4}{3r^3}\Big\{- K^2[CJ_6(\eta) - (A + \tfrac{5}{2}C)J_4(\eta) + \tfrac{2}{5}AJ_6(\eta)]$$
$$- K(r-1)[CJ_5(\eta) - (A + \tfrac{5}{2}C)J_3(\eta) + \tfrac{2}{5}AJ_5(\eta)]$$
$$- (r-1)^2\tfrac{2A}{5}J_4(\eta)\Big\} + \tfrac{4}{3}(2 - \frac{A}{r} - \frac{B}{r^3})J_4(\eta)$$
$$+\frac{4}{3K}\int_{r+1}^{\infty}\Big\{-\frac{1}{K\tau^2}[(A + \tfrac{5}{2}C)J_2(\tau) + CJ_4(\tau)]$$
$$- 2J_3(\tau)[1 - (\frac{1}{t} - \frac{2r^2}{5t^3})A + \frac{B}{t^3}]\Big\}dt, \qquad (20)$$

$$G_{K,4} = -\frac{4K}{r^2}[CJ_6(\eta) - (A + 4C)J_4(\eta) + \tfrac{3}{2}(A + \tfrac{5}{2}C)J_2(\eta)], \qquad (21)$$

$$F_{A,1} = \sin\chi\Big\{K\frac{r-t\cos\chi}{s^3}[CJ_3(\tau) - (A + \tfrac{5}{2}C)J_1(\tau)]$$
$$+ 2J_2(\tau)[(1 - \frac{A}{s} + \frac{B}{s^3})\cos\chi + (\frac{A}{2s} - \frac{3B}{2s^3})\frac{rt}{s^2}\sin^2\chi]\Big\}, \qquad (22)$$

$$F_{A,2} = \sin\chi\cos\chi\Big\{K\frac{r-t\cos\chi}{s^3}[CJ_4(\tau) - (A + \tfrac{5}{2}C)J_2(\tau)]$$
$$+ 2J_3(\tau)[(1 - \frac{A}{s} + \frac{B}{s^3})\cos\chi + (\frac{A}{2s} - \frac{3B}{2s^3})\frac{rt}{s^2}\sin^2\chi]\Big\}, \qquad (23)$$

$$F_{A,3} = \tfrac{1}{2}\sin^3\chi\Big\{K\frac{t}{s^3}[CJ_4(\tau) - (A + \tfrac{5}{2}C)J_2(\tau)]$$
$$+ 2J_3(\tau)[-(1 - \frac{A}{s} + \frac{B}{s^3}) + (-\frac{A}{2} + \frac{3B}{2s^3})\frac{r}{s^3}(r-t\cos\chi]\Big\}, \qquad (24)$$

$$F_{A,4} = \sin\chi\Big\{K\frac{r-t\cos\chi}{s^3}[CJ_5(\tau) - (A + 4C)J_3(\tau) + \tfrac{3}{2}(A + \tfrac{5}{2}C)J_1(\tau)]$$
$$+ 2[J_4(\tau) - \tfrac{3}{2}J_2(\tau)][(1-\tfrac{A}{s}+\tfrac{B}{s^3})\cos\chi + (\tfrac{A}{2s} - \tfrac{3B}{2s^3})\tfrac{rt}{s^2}\sin^2\chi]\Big\}, \quad (25)$$

$$F_{K,1} = \Sigma(s)J_1(\tau)\cos\psi + [J_3(\tau) - \tfrac{3}{2}J_1(\tau)]\Theta(s)\cos\psi$$
$$+ 2J_2(\tau)[Q_r(s)\frac{r\cos\psi-s}{t}\cos\psi + Q_\theta(s)\tfrac{r}{t}\sin^2\psi], \quad (26)$$

$$F_{K,2} = \Sigma(s)J_2(\tau)\cos\psi + [J_4(\tau) - \tfrac{3}{2}J_2(\tau)]\Theta(s)\cos\psi$$
$$+ 2J_3(\tau)[Q_r(s)\frac{r\cos\psi-s}{t}\cos\psi + Q_\theta(s)\tfrac{r}{t}\sin^2\psi], \quad (27)$$

$$F_{K,3} = \tfrac{1}{2}\Big\{\Sigma(s)\tfrac{t}{s}J_2(\tau) + [J_4(\tau) - \tfrac{3}{2}J_2(\tau)]\tfrac{t}{s}\Theta(s)$$
$$+ 2J_3(\tau)[Q_r(s)\frac{r\cos\psi-s}{s} - Q_\theta(s)\tfrac{r}{s}\cos\psi]\Big\}, \quad (28)$$

$$F_{K,4} = \Sigma(s)[J_3(\tau)-\tfrac{3}{2}J_1(\tau)]\cos\psi + [J_5(\tau)-3J_3(\tau)+\tfrac{9}{4}J_1(\tau)]\Theta(s)\cos\psi$$
$$+ 2[J_4(\tau)-\tfrac{3}{2}J_2(\tau)][Q_r(s)\frac{r\cos\psi-s}{t}\cos\psi + Q_\theta(s)\tfrac{r}{t}\sin^2\psi], \quad (29)$$

$$H_{K,1}=H_{K,4}= \frac{s\sin\psi}{t}, \quad H_{K,2} = \frac{s\sin\psi(r-s\cos\psi)}{t^2}, \quad H_{K,3} = (\frac{s\sin\psi}{t})^3, \quad (30)$$

$$J_n(\chi) = \int_0^\infty t^n\, e^{-t^2-\chi/t}\, dt, \quad (31)$$

$$\beta = \sin^{-1}(\tfrac{1}{r}), \quad \psi_0 = \cos^{-1}(\tfrac{1}{r}) + \cos^{-1}(\tfrac{1}{s}), \quad (32)$$

$$t_0 = r\cos\chi - \sqrt{1 - r^2\sin^2\chi}, \quad (33)$$

$$t = (r^2 - 2rs\cos\psi + s^2)^{1/2}, \quad (34)$$

$$s = (r^2 - 2rt\cos\chi + t^2)^{1/2}, \quad (35)$$

$$\tau = \frac{t}{K}, \quad \eta = \frac{r-1}{K}. \quad (36)$$

The integral equations are to be solved under the condition that the unknown functions should vanish as $\eta \to \infty$. The unknown constants A, B and C are to be determined with the solutions. Before showing the numerical results, we here give the formula of the drag acting on the sphere and the results of the free molecular flow. The drag may be calculated by taking a large control volume in the Stokes region enclosing the sphere and by applying the conservation law of the momentum to the volume. The drag coefficient is given by

$$C_D = \frac{\text{Drag}}{\frac{1}{2}mn_0Q_0{}^2(\pi a^2)} = 4\,\frac{K}{S}\,A. \tag{37}$$

It may be noted that the constant temperature of the sphere T_w does not contribute to the force on the sphere, because this is taken to be constant and hence induces the symmetric temperature and velocity fields around the sphere.

The free molecular flow is easily evaluated and the results are

$$\sigma = -\frac{1}{\sqrt{\pi}r^2}\cos\theta, \tag{38}$$

$$\omega = -\frac{1}{3\sqrt{\pi}r^2}\cos\theta, \tag{39}$$

$$q_r = \frac{S}{2}\{1 + \frac{(r^2-1)^{3/2}}{r^3}\}\cos\theta, \tag{40}$$

$$q_\theta = -\frac{S}{4}\{2 + \frac{3\sqrt{r^2-1}}{r} - \frac{(r^2-1)^{3/2}}{r}\}\sin\theta. \tag{41}$$

4. NUMERICAL RESULT

The method of the numerical calculation is the same as is in Refs.13 and 14 . We have carried out the calculation for the Knudsen number covering from the nearly free molecular flow to the near continuum flow. Some of the result are shown here.

The distributions of the flow velocity $\underline{Q}$, temperature Θ and density Σ in the kinetic region for typical four cases of the

Knudsen number K=0.01,0.1, 1 and, 10 are plotted in Figs. 1, 2, 3, and 4 respectively. The results by Onishi[6] for small K are also marked in Fig.1 when K is taken to be 0.01 in his result. It will be seen that our result agrees well with Onishi's. The distribution of the flow velocity $\underline{q}$, temperature ω and density σ at K=50 are shown in Fig.5. The cross in the same figure shows the distributions of the free molecular flow.

The varation of the flow velocity q_r at the surface versus K, which corresponds to the evaporation or condensation velocity, is shown in Fig.6. The straight dotted line for small K is taken from Onishi's result, while the horizontal dotted line for large K shows the value of free molecule's.

Numerical values of the constsnts A, B, and C, which are involved in the flow velocity, temperature and density, are listed in Table I. The value A in Table I is also included in the formula of the drag (see Eq.(37)). The corresponding values of A ,B and C by Onishi[6] are shown in the same table. The agreement of two results are quite good.

The drag coefficient C_D from Eq.(37) is plotted in Fig.5. The square on the right vertical line shows the drag coefficient of the free molecular flow,[9] and the circles are taken from the paper by Beresnev et al..[11] The dotted line represents the drag coefficient of the solid sphere for which there is neither condensation nor evaporation and whose data are taken from Lea and Loyalka,[5] and supplemented by the present author for several values of large K. The drag reduction is clearly seen if the evaporation and condensation takes place at the surface. It is also noticed that the drag reduction is larger for larger Knudsen numbers.

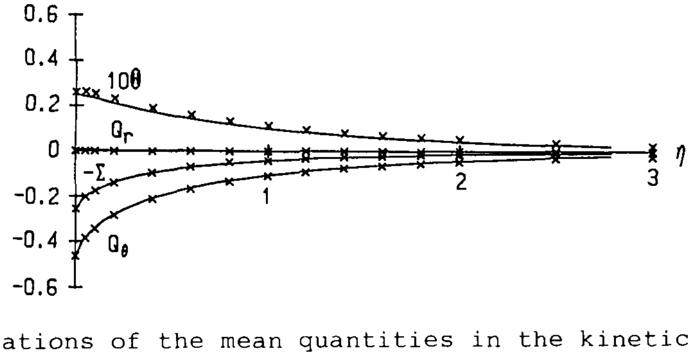

Fig.1. Variations of the mean quantities in the kinetic region at K=0.01. x: Onishi.[6]

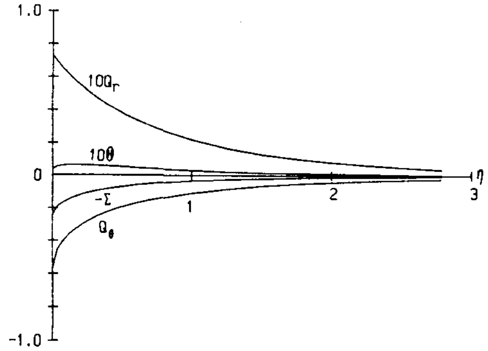

Fig.2. Variations of the mean quantities in the kinetic region at K=0.1.

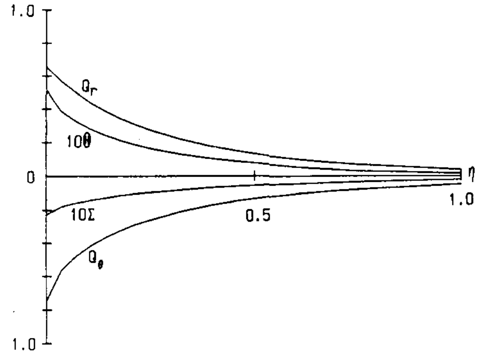

Fig.3. Variations of the mean quantities in the kinetic region at K=1.0.

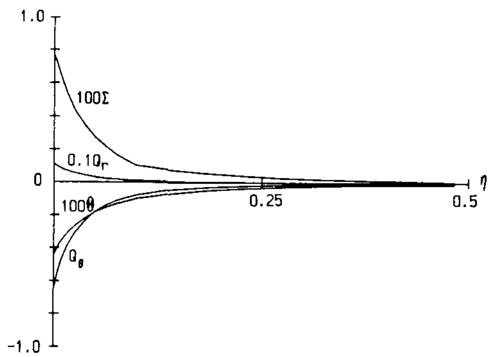

Fig.4. Variations of the mean quantities in the kinetic region at K=10.0.

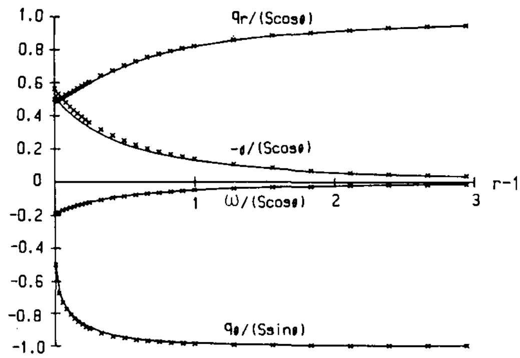

Fig.5. Distribution of the mean quantities at K=50.0. x: free molecular flow.

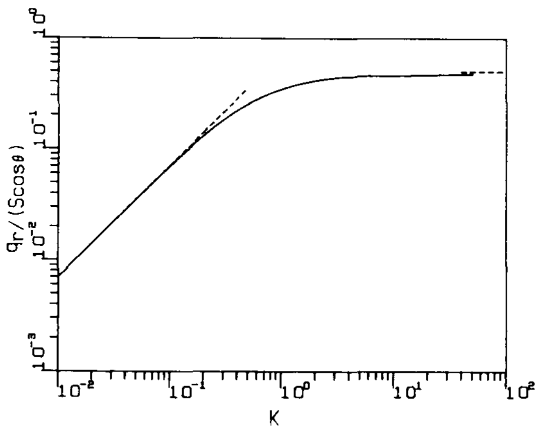

Fig.6. Radial flow velocity at the surface versus K. ----- (K<<1): Onishi,[6] -----(K>>1): free molecular flow.

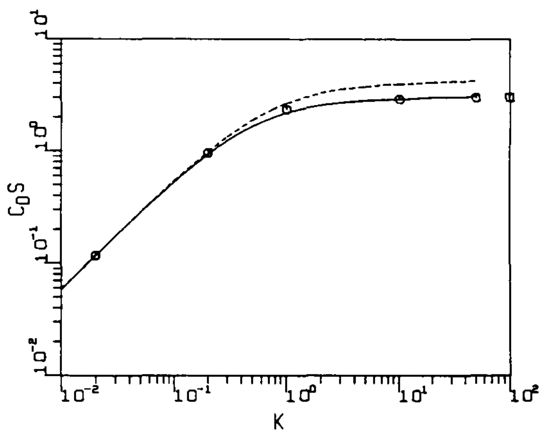

Fig.7. Drag coefficient versus K. □:free molecular flow, ○: Beresnev et al,[11] -----:solid sphere.

Table I Values of A, B and C.

K	A	A*	B	B*	C	C*
0.01	1.481	1.4812	0.4883	0.48827	-0.3058	-0.31431
0.05	1.408	1.4062	0.4412	0.44137	-0.2765	
0.1	1.320	1.3124	0.3821	0.38275	-0.2510	
0.2	1.164	1.1248				
0.3	1.033	0.93718				
0.5	0.8331	0.56197				
0.75	0.6645					
1	0.5500		-0.7613		-0.1157	
2	0.3213					
3	0.2254					
5	0.1410					
7.5	0.09575					
10	0.07246		-12.07		-0.01380	
20	0.03753					
30	0.02509					
50	0.01539					

* Onishi.[6]

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