Memoirs of the Faculty of Engineering, Okayama University, Vol.29, No. 2, pp.49-55, March 1995

Evaluation of Eigenmodes of Dielectric Waveguides by a Numerical Method Based on the BPM

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(Received January 27 , 1995)

SYNOPSIS

For weakly guiding dielectric waveguides, the eigenmode field distributions are calculated numerically with a simple algorithm. In this numerical method, the transverse sampling space can be chosen arbitrarily, and hence a narrow waveguide can be analyzed. The field satisfying scalar wave equation is expressed by the discrete Fourier transform and the mode eigenvalues and eigenfunctions are calculated by solving an eigenvalue equation numerically. The validity of this method is checked for 2-D waveguides having step and parabolic or square index distributions. It is found that for the well guided TE modes of the slab waveguide, the accuracy of this method is remarkably good, but some discrepancies are found if the mode is near cut off. In the problems where the normalized guide index b is small, caution should be taken in applying the results of this numerical method.

1. INTRODUCTION

Recently, there has been considerable interest in the evaluation of light propagation characteristics of dielectric waveguides used for the optical integrated circuits. To get reliable results, it is necessary that the field distribution and the propagation constant should be the mode eigenfunction and eigenvalue of the waveguide, respectively. For some known refractive index profiles of a 2-D waveguide [1], the reduced wave equation can be solved analytically. However, these index profiles are not same as the index profiles of a practical dielectric waveguide. For the realistic index distribution of a guide, it is very difficult to get an analytical solution of the reduced wave equation. So various numerical methods have been developed to solve the reduced wave equation, and hence to find the mode characteristics.

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One of the numerical methods is the beam propagation method (BPM) by which the mode eigenfunctions and eigenvalues are calculated [2]. But this method needs vast computer memory for the repeated calculation of the complex field up to a reasonable propagation distance. Another improved numerical method is developed by expressing the BPM equation into the coupled mode equation, which is solved numerically as an eigenvalue problem [3]. In this paper, we describe a numerical method which is simpler than the method given in Ref. [3]. The present method is based on the formulation of the BPM. All the fundamental approximations of this method are same as that of the BPM, but in this method the transverse sampling space can be chosen arbitrarily. The scalar Helmholtz equation is first approximated to get Fresnel equation [2], whose field is expanded by the discrete Fourier transform. The scalar Fresnel equation is then expressed in the form of an eigenvalue equation which is solved for the mode eigenvalues and eigenfunctions.

To show the exactness of the present numerical method, the eigenmodes of a narrow waveguide and also of a wide waveguide with step and parabolic or square index distributions are calculated. The obtained results are compared with that of the exact analytical ones. It is found that both results agree well for the lower order eigenmodes, but for higher order eigenmodes some discrepancy between the two results are found near the waveguide boundaries. According to Ref. [3] these discrepancies are due to the virtual boundaries placed at the waveguide edges for the convenience of the numerical calculation. In this paper, we make it clear that this conjecture is not true. The above disagreements are occurred when the normalized guide index (b) is small. For simplicity, we limit our discussion for the TE modes of a 2-D waveguide. In section 2, the derivation of the eigenvalue equation is given, and in section 3, we compare the numerical results with the analytical ones.

2. DERIVATION OF THE EIGENVALUE EQUATION

Let us consider that the propagation of light through a dielectric slab waveguide is in the z-direction. The field of this waveguide is taken uniform in the y direction ($\partial/\partial y = 0$) and is confined in the x-direction only. The TE modes of this waveguide consist of the field components E_y, H_x and H_z.

Considering the medium is isotropic, source-free and inhomogeneous, the wave equation for the TE modes can be expressed as

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} + k_0^2\varepsilon_r(x,z)\psi = 0 \tag{1}$$

where $\psi(x,z) = E_y(x,z)$, ε_r is the relative dielectric constant of the medium, and $k_0 = \omega^2\mu_0\varepsilon_0 = 2\pi/\lambda_0$; ω is the angular frequency, and k_0, λ_0, μ_0 and ε_0 are the free space wavenumber, wavelength, magnetic permeability and dielectric constant, respectively. In deriving the above equation we have considered the time dependence of the field as exp(jωt). Equation (1) is also called Helmholtz equation. The relations of E_y with the other field components (H_x and H_z) of the TE modes are found from Maxwell's equations. Our aim is to solve Eq. (1) with an approximated numerical method, which is similar to the BPM.

Since we are interested in the calculation of the eigenmodes of the waveguide, we assume that the structure of the waveguide is uniform in the z-direction. In that case the relative dielectric constant ε_r will be a function of only one coordinate axis x and is written as $\varepsilon_r = n^2(x)$. If δ is the maximum index change of the guide, then a general expression of the index distribution is given by

$$n(x) = n_s + \delta f(x) \tag{2}$$

where $f(x)$ is the index distribution function. The parameter δ is, generally, very small in comparison with n_s. Therefore, we can write

$$n^2 \cong n_s^2 + 2n_s\delta f(x). \qquad (3)$$

Let us consider that one type of solution of Eq. (1) may be expressed as

$$\psi(x,z) = \phi(x,z)\exp\{-\mathrm{j}k_0 n_s z\} \qquad (4)$$

where n_s is the refractive index of the substrate. Substituting Eq. (4) into Eq. (1), and using the approximation that the change of the refractive index is very slight over the distance of one wavelength, we get the parabolic or Fresnel wave equation as [2]

$$\frac{\partial \phi'}{\partial z} = \frac{-\mathrm{j}}{2k_0 n_s}\left[\nabla_t^2 + k_0^2\left\{n^2 - n_s^2\right\}\right]\phi' \qquad (5)$$

where $\nabla_t^2 = \partial^2/\partial x^2$ and ϕ' represents the field distribution of the above wave equation which is obtained by neglecting the term containing the double derivative of ϕ with respect to z. So hereafter, a prime will be used for the parameters belong to Fresnel equation.

Let the eigensolutions of Eq. (1) and Eq. (5) are, respectively

$$\psi(x,z) = t_m(x)\exp(-\mathrm{j}k_0 N_{\text{eff}} z) \qquad (6)$$

and

$$\phi'(x,z) = t'_m(x)\exp(-\mathrm{j}k_0 n'_{\text{eff}} z) \qquad (7)$$

where N_{eff} and n'_{eff} are the effective indices found from Eqs. (1) and (5), respectively. The effective index N_{eff} found from Helmholtz equation (Eq. (1)) is used to define the propagation constant $\beta = k_0 N_{\text{eff}}$. The effective index n'_{eff} found from Fresnel equation is related to N_{eff} by the relation [2]

$$n'_{\text{eff}} = (N_{\text{eff}}^2 - n_s^2)/(2n_s). \qquad (8)$$

Similarly, the field distributions of Fresnel equation is related to that of Helmholtz equation by the relation [2]

$$t'_m(x) = t_m(x). \qquad (9)$$

Thus it is clear that the solutions of Helmholtz and Fresnel equation will provide us with the same eigenmode field distributions, but their eigenvalues are related by Eq. (8).

Substituting Eq. (7) in Eq. (5), we get

$$n'_{\text{eff}}\phi' = \frac{1}{2k_0^2 n_s}\left[\nabla_t^2 + k_0^2\left\{n^2 - n_s^2\right\}\right]\phi'. \qquad (10)$$

The above equation can again be modified to the following form by substituting Eq. (3) in it

$$\frac{n'_{\text{eff}}}{\delta}\phi' = \left[\frac{\nabla_t^2}{2k_0^2 n_s\delta} + f(x)\right]\phi'. \qquad (11)$$

The approximated field ϕ' is now expanded by the discrete Fourier transform as

$$\phi'(x_i,z) = \sum_n \Phi'_n(z)\exp(\mathrm{j}\nu_n x_i) \qquad (12)$$

$$\Phi'_n(z) = \frac{1}{N}\sum_i \phi'(x_i,z)\exp(-\mathrm{j}\nu_n x_i) \qquad (13)$$

where $i,n = -(N/2-1),\ldots,-1,0,1,\ldots,N/2$, $x_i = Di/N$, and the transverse wavenumber $\nu_n = 2\pi n/D$; N is the total sampling points, and D is the length of the calculated region. In matrix form, Eqs. (12) and (13) are written as

$$\begin{aligned}\boldsymbol{\phi}' &= \boldsymbol{F}^{-1}\boldsymbol{\Phi}' \\ \boldsymbol{\Phi}' &= \boldsymbol{F}\boldsymbol{\phi}' \end{aligned} \qquad (14)$$

where $\boldsymbol{F} = 1/N \cdot [F_{ni}] = 1/N \cdot [\exp(-\mathrm{j}\nu_n x_i)]$ and $\boldsymbol{F}^{-1} = [F_{in}^{-1}] = [\exp(\mathrm{j}\nu_n x_i)]$. Substituting Eq. (14) in Eq. (11), we get

$$\frac{n'_{\text{eff}}}{\delta}\phi' = \left[q\boldsymbol{F}^{-1}\boldsymbol{K}\boldsymbol{F} + \boldsymbol{R}\right]\phi' \qquad (15)$$

where, matrices $\boldsymbol{K}$ and $\boldsymbol{R}$ are the two diagonal matrices with elements $\nabla_t^2 = \partial^2/\partial x_i^2 = -\nu_i^2$ and $f(x_i)$, respectively, and q is equal to $1/(2k_0^2 n_s\delta)$.

Let us define the approximated normalized guide index b' as

$$b' = \frac{N_{\text{eff}}^2 - n_s^2}{n_0^2 - n_s^2} \cong \frac{n'_{\text{eff}}}{\delta} \qquad (16)$$

where $n_0 = n(x = 0)$ is the core index at $x = 0$. The value of b' may vary from 0 to 1. Using the above equation, we can write Eq. (15) as an eigenvalue equation, i.e.

$$b'\phi' = \boldsymbol{P}\phi' \tag{17}$$

where matrix $\boldsymbol{P} = q\boldsymbol{F}^{-1}\boldsymbol{K}\boldsymbol{F} + \boldsymbol{R}$. Without calculating the beam propagation in the z-direction [2], we can calculate the mode eigenfunctions and the eigenvalues at a plane $z = 0$ by solving Eq. (17). Since we have used the slow index variation approximation, the wave equation of TM modes can also be made similar to Eq. (1) for H_y ignoring the term containing the derivative of ε [5]. So the solutions for the TM modes will be similar to the TE modes.

3. NUMERICAL EXAMPLES

A. Eigenmodes of Step Index Waveguide

The present numerical method is first applied to a symmetric 2-D waveguide with step index distribution as shown by the insert of Fig. 1. The half- and full-width of the waveguide are denoted by d and T, respectively. Depending on the waveguide width, two types of waveguides are analyzed. For a narrow waveguide (where only fundamental mode exits) T is taken 2 μm, and for a wide waveguide (where three modes exist) T is chosen 8 μm. The normalized frequency V of the narrow waveguide is taken 2 and that of the wide waveguide is chosen 8. The value of $f(x)$ in Eq. (9) is 1 in the core and 0 in the cladding region. The other parameters are $n_s = 1.0$, $\lambda_0 = 1$ μm, $D = 40$ μm and $N = 256$. The maximum index change δ can then be calculated from the relation $V = k_0 T(n_f^2 - n_s^2)^{1/2}$, where $n_f = n_0$ is the core index for step index guide. In our case δ is found to be 0.012665148. The analytical eigenmode field distributions are found from the solutions of the wave equation as given in Ref. [6].

The eigenmode field distribution of the narrow waveguide obtained by this numerical method is plotted in Fig. 1(a) and is shown by the dotted curve. In the wide waveguide, 3 guided modes are found and their field distributions are plotted in Fig. 1(b) and are shown

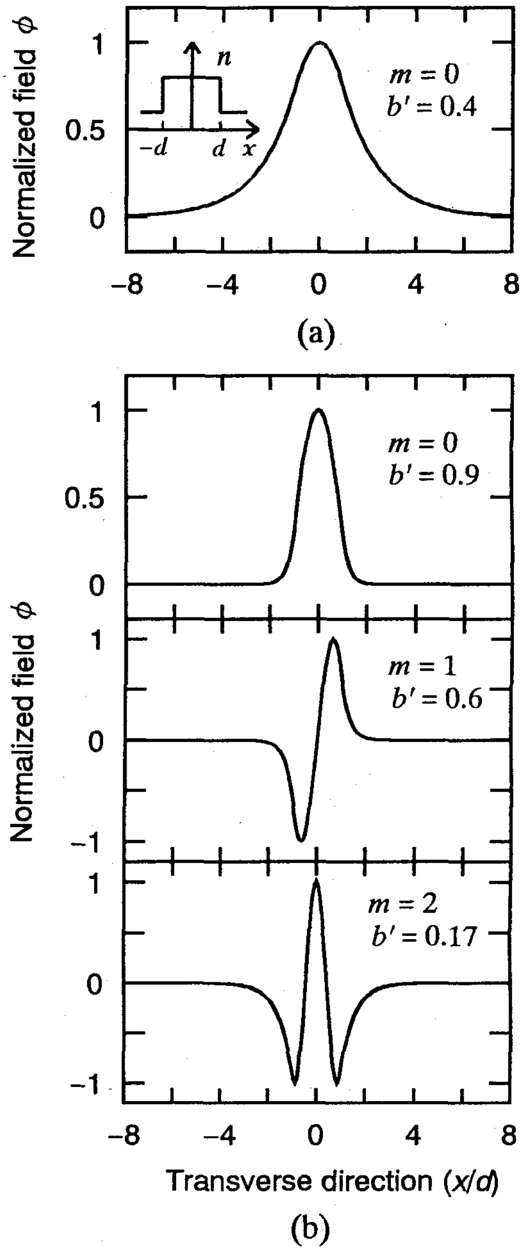

Fig. 1 Mode field distributions obtained by the numerical (dotted curves) and analytical methods (solid curves) with step index profile; (a) narrow waveguide and (b) wide waveguide.

by the dotted curves. The analytical eigenmode field distributions of the narrow as well as the wide waveguide are also plotted in Fig. 1 by the solid curves. It is seen that the dotted curves are completely coincide with the solid curves for both types of waveguides. The well agreement of both solutions confirms the validity of Eq. (7).

B. Eigenmodes of Truncated Square Index Waveguide

As a second example, the present numerical method is applied to a symmetric slab waveguide with truncated square index distribution (Fig. 2) which is expressed as [1]

$$n^2(x) = \begin{cases} n_0^2(1 - x^2/x_0^2) & |x| \le d \\ n_s^2 & |x| \ge d \end{cases} \tag{18}$$

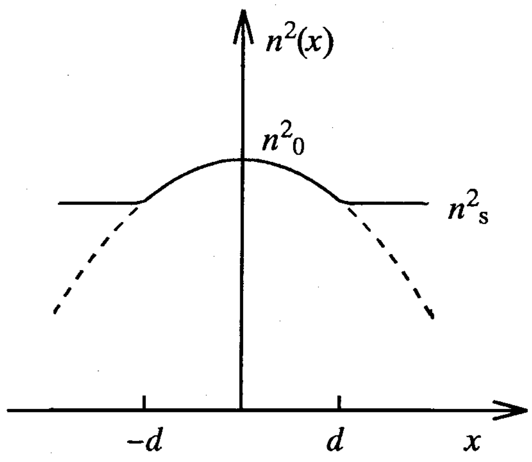

Fig. 2 The truncated square index distribution shown by the solid curve for which the present calculation is done and the typical infinite square index profile shown by the dashed curve.

The parameter x_0 is selected such that the index at a distance $|x| = d$ will be equal to n_s. All the parameters of the square index case, except T and V, are kept same as those of the step index case explained in the Example A. For a narrow waveguide (single-mode waveguide) T is 4 μm and V is 4, and for a wide waveguide (multi-mode waveguide) T is chosen 12 μm and V is taken 12.

The exact eigenmode field distributions are calculated by the mode matching method. In the region $|x| \le d$ the exact field is expressed by the parabolic cylindrical function and outside this region the field will decay exponentially. By matching the tangential field components of E_y and H_z at the boundary $|x| = d$, the field amplitude and its propagation constant are found. For a well guided mode, the order of the parabolic cylindrical function is very close to some integer value. In that case the parabolic function can be approximated by Hermite-Gauss function.

Considering the simplicity of the analytical solution, we used Hermite-Gauss function to express the field in the guide. Thus, within the region $|x| \le d$ the exact field is expressed by Hermite-Gauss function as [1],

$$\psi(x) = \mathrm{H}_m(\sqrt{2}x/w)\exp(-x^2/w^2) \tag{19}$$

where m is the mode number, $\mathrm{H}_m(\xi)$ is the mth order Hermite function, and w is called the *beam radius* given by

$$w^2 = \lambda_0 x_0/(\pi n_0). \tag{20}$$

The parameter w indicates the degree of confinement of the fundamental mode. In the region $|x| \ge d$, the field is expressed as $A\exp(-\gamma|x|)$. The amplitude A and the propagation constant γ are found by matching the tangential field components of E_y and H_z at the boundary $|x| = d$ and they are found as

$$A = \mathrm{H}_m(\sqrt{2}d/w)\exp(-d^2/w^2)\exp(\gamma d) \tag{21}$$

$$\gamma = \frac{2d}{w^2} - \frac{2\sqrt{2}\cdot m}{w}\frac{\mathrm{H}_{m-1}(\sqrt{2}d/w)}{\mathrm{H}_m(\sqrt{2}d/w)} \tag{22}$$

In Fig. 3, both the analytical (solid curves) and the numerically (dotted curves) obtained eigenmode field distributions are plotted. The mode field distributions of the single-mode waveguide and the multi-mode waveguide are shown in Fig. 3(a) and in Fig. 3(b), respectively. It is seen that the numerical results agree well with the analytical ones for the lower order modes of a multi-mode waveguide, and is consistent with Eq. (7). But some disagreements of these two results are found if the mode belong to a square index waveguide. For the fundamental mode of a single-mode waveguide as well as for a higher order mode of a multi-mode waveguide, these discrepancies are due to the beam radius w and the normalized guide index b'.

Using Eq. (20), the fundamental mode beam radius w of the single-mode and the multi-mode waveguide are found to be 2.0 μm and 3.46 μm, respectively. The ratios of w/d for the single- and multi-mode waveguides are 1.0 and 0.58, respectively. Since w indicates the degree of confinement of the fundamental mode, smaller value of w in comparison to d means better confinement of fundamental mode. Thus the confinement of the fundamental mode field in the multi-mode waveguide is better than that in the single-mode waveguide. This description is clearly visible in the field distributions for $m = 0$ shown in Fig. 3(a) and in Fig. 3(b). Since the guidance of the fundamental mode is weak in the single-mode waveguide, some errors are found in the numerical calculations. With the increase of the mode number the confinement of the modal power decreases. So for higher order modes of the multi-mode waveguide, the numerical results have also errors near the boundary of guide. For $m = 2$, such field distribution is shown in Fig. 3(b).

For the well confined modes b' is large (for example, in our obtained results shown in Fig. 3(b), b' is 0.8 when $m = 0$) and for the weakly confined modes which are near cut off b' is small (for example, in our obtained results shown in Fig. 3(b), b' is 0.16 when $m = 2$). Thus with the increase of mode number m, the mode field distributions in the guide spread out further from the guide axis and the numerical results are no longer a good approximation to the exact ones.

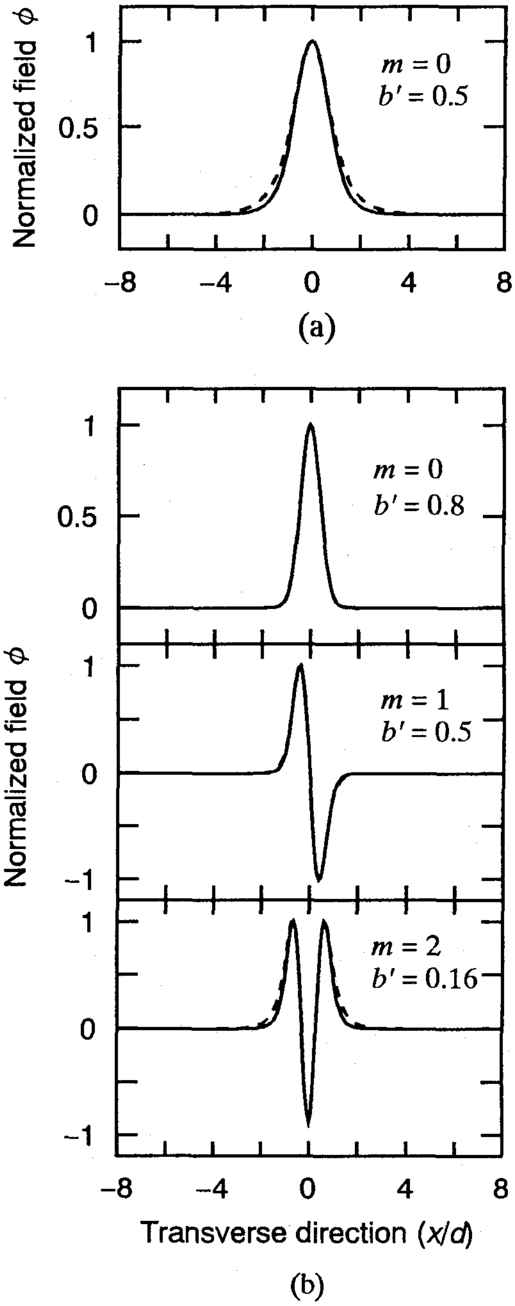

Fig. 3 Mode field distributions for the truncated square index profile obtained by the numerical (dotted curves) and analytical calculations (solid curves); (a) single-mode waveguide and (b) multi-mode waveguide.

Furthermore, the gradient of index distribution near the region $|x| = d$ is relatively larger than that near the waveguide axis. So the value of matrix $\boldsymbol{P}$ near $|x| = d$ is not as accurate as that near the waveguide axis. Thus in calculating the higher order mode fields at the vicinity of the waveguide edges, some error may arise in the numerical results. The authors of Ref. [3] have claimed that the above disagreements are occurred due to the virtual boundaries placed at the core and cladding interfaces of the multi-mode waveguide. But with the present calculation these discrepancies are present even the virtual boundaries are moved to a distance equal to $3d$. From the above discussion it is clear that the obtained disagreement between the analytical and numerical solutions are not due to the virtual boundaries, but due to the effects of b', w and ∇n. Since the present numerical method is based on the BPM, it can be inferred that the stationary mode eigenfunctions obtained the BPM [2] is not accurate enough if the normalized guide index b' is small.

4. CONCLUSIONS

The eigenmode field distributions of the slab optical waveguide with step and truncated square index distributions are evaluated numerically with a simple algorithm. For the well guided TE modes of the slab waveguide, the accuracy of this method is remarkably good, but some discrepancies are found if the mode is near cut off. The basic approximations of this numerical method are same as that of the BPM, but the transverse sampling space of the present calculation can be chosen arbitrarily. Using slow index variation approximation, the present numerical method can also be used for the analysis of the TM modes [5].

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