

Abductive Proof Procedure with Adjusting Derivations for General Logic Programs

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(Received December 7, 2000)

In this paper, we formulate a new integrity constraint in correlation with 3-valued stable models in an abduction framework based on general logic programs. Under the constraint, not every ground atom or its negation is a logical consequence of the theory and an expected abductive explanation, but some atom may be unspecified as a logical consequence by an adjustment. As a reflection of the integrity constraint with an adjustment, we augment an adjusting derivation to Eshghi and Kowalski abductive proof procedure, in which such an unspecified atom can be dealt with.

1 Introduction

Abductive logic programming has been established primarily for the theory of general logic programs by means of negation as failure, as its refinements are summed up in [12]. The original Eshghi and Kowalski abduction framework, a triplet (P, Ab, I) of a general logic program (theory) P , a set of abducibles Ab and an integrity I , was captured by the 2-valued stable model semantics in the sense that the abductive explanation is in a close relation with a 2-valued stable model of a program (theory). The integrity constraint I requires that any ground atom or its negation is a logical consequence of the theory and abducibles, but both are not. That is, if $P \cup \Delta$ satisfies the constraint I for an abductive explanation Δ , then $M = \{a \mid \neg a \notin \Delta\}$ is a 2-valued stable model of P , and if M is a 2-valued stable model of P , then $P \cup \Delta$ satisfies the constraint I for $\Delta = \{\neg a \mid a \notin M\}$. However, Eshghi and Kowalski abductive proof procedure (E-K procedure, for short) is not in general sound with respect to the 2-valued stable model semantics [7]. The correctness of E-K procedure is guaranteed in 3-valued logic in that it is sound with respect to a 3-valued stable model [4, 6, 13].

We are motivated to formulate an integrity constraint under which an abductive explanation is related to a 3-valued stable model. Owing to the 3-valued logic structure, an abductive adjustment, as well as an abductive explanation, is taken into shape so that the adjustment allows the atoms neither to be in an abductive explanation, nor to be a logical consequence

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of the theory and abducibles. In a more formal form, we intend an integrity constraint I_t for a program P with its Herbrand base B_P so that, for an adjustment Γ and an abductive explanation Δ , if $P \cup \Gamma \cup \Delta$ satisfies I_t , then $(B_P - (\Gamma \cup \Delta), \Delta)$ is a 3-valued stable model and if (T, F) is a 3-valued stable model, then $P \cup (B_P - (T \cup F)) \cup F$ satisfies I_t .

As a next step, in order to extract not only an abductive explanation, but also an adjustment, we augment an adjusting derivation to the original E-K procedure, in which the enumeration of the adjustment is made. By such an augmentation, the abductive proof procedure is defined as consisting of succeeding, finitely failing and adjusting derivations. As regards a merit of the augmentation, we regard all the three derivations as possibly initial phases. Especially an adjusting derivation can be a prerequisite for an expected succeeding derivation, since it detects and enumerates an adjustment before getting abducibles.

We must at the least examine the soundness of the augmented abductive procedure with respect to the newly presented integrity constraint, before confronting the difficult problem of the completeness of the abductive proof procedure. Because the integrity constraint is expected to be related to a 3-valued stable model, the augmented procedure with an adjusting derivation is to be proved sound with respect to a 3-valued stable model. Concerning the 3-valued stable model, the alternating fixpoint semantics ([1, 9]) is adequate in relation to the constraint. Because the abductive explanation, as well as the logical consequence of the theory and abducibles, is easy to be denoted by the fixpoint theory approach, though the well-founded model [8] is an alternative. As well, the adjustment may be described by using the alternating fixpoint techniques. We so far show a relation between the integrity constraint and the 3-valued stable model through the alternating fixpoint.

On the other hand, rigid discussions regarding the soundness of the augmented abductive proof procedure are necessary by induction on the length of derivations as natures of abductive proof procedures, but may not be free from the procedural aspects even in terms of the alternating fixpoints.

This paper is organized as follows. Section 2 summarizes technical terms and basic results in model theory for general logic programs. In Section 3, we have an abduction framework in which a new integrity constraint is introduced so that we may have a natural relation between the constraint and the 3-valued stable model. In Section 4, an abductive proof procedure, with an adjusting derivation augmented, is given and its soundness is shown. In Section 5, technical merits are stated.

2 Model Theory for General Logic Programs

2.1 General Logic Programs

A general logic program is a set of clauses of the form:

$$A \leftarrow L_1 \dots L_n \ (n \geq 0),$$

where A is an atom, and L_i are literals. A is called the head of the clause, and $L_1 \dots L_n$ its body. A normal goal is an expression of the form:

$$\leftarrow L_1 \dots L_n,$$

where L_i are literals. If the clause contains any negation in its body, then it is said a definite clause. A clause involving no variables is called a ground clause. As well an expression without any variables is said a ground expression.

For a general logic program P , its Herbrand base, constructed by any predicate symbols and function symbols in it, is denoted B_P . Let $(B_P)^*$ be $\{p^*(t_1, \dots, t_n) \mid p(t_1, \dots, t_n) \in B_P\}$.

Definition 2.1 For $S \subseteq B_P$ let $S^* = \{a^* \in (B_P)^* \mid a \in S\}$. For $T \subseteq (B_P)^*$ let $T^+ = \{a \in B_P \mid a^* \in T\}$.

2.2 Model Theory

A 2-valued Herbrand interpretation is a subset of the Herbrand base, for which a ground atom included in the interpretation is evaluated as true, and a ground atom not included in it as false. In the context of the present paper, a (Herbrand) model of a general logic program is a (Herbrand) interpretation in which any clause of the program is true. The following transformation associated with a set of definite clauses is often made use of.

Definition 2.2 For a set of definite clauses P , $T_P : 2^{B_P} \rightarrow 2^{B_P}$ is defined to be

$$T_P(I) = \{A \in B_P \mid \exists A \leftarrow A_1 \dots A_n \in \text{ground}(P) \text{ and } \{A_1, \dots, A_n\} \subseteq I\},$$

where $\text{ground}(P)$ is the set of all ground clauses obtained from clauses of P .

Now we take interpretations and models in 3-valued logic, where the truth value **t** (the truth), **f** (the falsehood) and **u** (the undefined) are used. As in [16], the logical connectives follow the truth tables.

$\neg$		$\wedge$	t	u	f	$\vee$	t	u	f
t	f	t	t	u	f	t	t	t	t
u	u	u	u	u	f	u	t	u	u
f	t	f	f	f	f	f	t	u	f

$\leftarrow$	t	u	f	$\leftrightarrow$	t	u	f
t	t	t	t	t	t	f	f
u	f	t	t	u	f	t	f
f	f	f	t	f	f	f	t

A 3-valued Herbrand interpretation, as in [16], is defined:

Definition 2.3 Given a general logic program P , a 3-valued Herbrand interpretation is a pair $\langle T, F \rangle$, where $T, F \subseteq B_P$ such that $T \cap F = \emptyset$.

A 3-valued model of a general logic program is a 3-valued Herbrand interpretation in which the program is true. Partial orders on the family of 3-valued Herbrand interpretations of a given general logic program P are given as in [11, 16].

Definition 2.4 Let $I = \langle T, F \rangle$, and $J = \langle T', F' \rangle$ be 3-valued Herbrand interpretations of a general logic program P . We define $\leq_t$ and $\leq_k$ to be

$$\begin{aligned} I \leq_t J &\Leftrightarrow T \subseteq T' \text{ and } F' \subseteq F, \text{ and} \\ I \leq_k J &\Leftrightarrow T \subseteq T' \text{ and } F \subseteq F'. \end{aligned}$$

t , f and u are used as atoms which are regarded as true, false and undefined, respectively, in any interpretation. $\models$ stands for the logical consequence relation in 2-valued logic, while $\models_3$ for that in 3-valued logic. In this paper, $\Gamma \models_3 F$ means that F is true in any 3-valued Herbrand interpretation making Γ evaluated as true.

As introduced in [16], the 3-valued stable model is defined as follows:

Definition 2.5 Let P be a general logic program and $I = \langle T, F \rangle$ its 3-valued Herbrand interpretation. P/I is a set of clauses obtained by implementing the following procedures for P :

- (1) Delete the clause whose body involves $\neg A$ such that $A \in T$.
- (2) Replace all the (negative) literals $\neg A$ by u , if $A \notin T \cup F$, in the remaining clauses.
- (3) Delete all the (negative) literals $\neg A$ in the still remaining clauses.

Let the least Herbrand model of P/I , with respect to $\leq_t$, be denoted $\Lambda(P/I)$.

Definition 2.6 Assume a general logic program P . A 3-valued Herbrand interpretation I is a 3-valued stable model iff $\Lambda(P/I) = I$.

2.3 Relation of Alternating Fixpoint with 3-Valued Stable Model

As in [9], there is a mapping whose fixpoint is closely related with the well-founded model [8]. The well-founded model is a least 3-valued stable model with respect to $\leq_k$ [17]. [5] discusses properties of the well-founded model. The relation between the 3-valued stable model and the fixpoint of the mapping is discussed in [22]. Following [1, 9], we have some mappings:

Definition 2.7 Let F be a 2-valued Herbrand interpretation, and P a general logic program. P^F is defined by means of the following procedures:

- (1) Let $A \leftarrow B_1 \dots B_n \bar{D}_1 \dots \bar{D}_m$ be in P^F for each $A \leftarrow B_1 \dots B_n \neg D_1 \dots \neg D_m \in \text{ground}(P)$, where $\bar{D}_i = \bar{p}(t_1, \dots, t_k)$ iff $D_i = p(t_1, \dots, t_k)$.
- (2) Let $A \leftarrow$ be in P^F for $A \in F$.

We define a mapping $S_P : 2^{B_P} \rightarrow 2^{B_P}$ to be $S_P(F) = B_P \cap \text{lfp}(T_{P^F})$, where $\text{lfp}(T_{P^F})$ is a least fixpoint of T_{P^F} . Let $A_P(F) = S_P(\overline{S_P(F)})$, where the overline stands for the complement with respect to B_P .

Based on the result in [22], we have:

Theorem 2.8 For a general logic program P and its 2-valued Herbrand interpretation F , $F = A_P(F)$ and $F \cap S_P(F) = \emptyset$ iff $\langle S_P(F), F \rangle$ is a 3-valued stable model of P .

3 New Integrity Constraint in Abduction Framework

In a new integrity constraint of an abduction framework, we prepare an auxiliary means, called an adjustment, to let intractable (ground) atoms be interpreted unspecified as a logical consequence of the theory and abducibles. This seems natural as far as we consider just the integrity constraint in 3-valued logic. By means of the constraint, any (ground) atom is evaluated as a logical consequence of the theory, the abducibles and adjustments to be true, false or undefined. With the new integrity constraint of an abduction framework, we will have an augmented abductive proof procedure on the basis of Eshghi and Kowalski procedure [7] (E-K-procedure, for short) in Section 4. From now on, the negation is sometimes replaced by “*” for the same treatment as in [12].

3.1 Abduction and Stable Model

An abduction framework [7] under some integrity constraint can be described by a 2-valued stable model [10].

Definition 3.1 An abduction framework based on a general logic program is a triplet $\langle P^*, Ab^*, I^* \rangle$, where:

(1) P^* is a set of definite clauses obtained by substituting $p^*(t_1, \dots, t_k)$ for $\neg p(t_1, \dots, t_k)$ in P .

(2) $Ab^* = (B_P)^*$ is a set of abducibles.

(3) I^* is the set of all integrity constraints of the form: $\forall \bar{x} \neg [p(\bar{x}) \wedge p^*(\bar{x})]$ and $\forall \bar{x} [p(\bar{x}) \vee p^*(\bar{x})]$. (Note that $\bar{x}$ stands for a tuple of variables, and p is a predicate symbol in P .)

For $\Delta \subseteq Ab^*$, we say that $P^* \cup \Delta$ satisfies I^* if for any $s \in B_P$

$$P^* \cup \Delta \not\models s \wedge s^*, \text{ and } P^* \cup \Delta \models s \text{ or } P^* \cup \Delta \models s^*.$$

If $P^* \cup \Delta$ satisfies I^* , then $\{a \mid a^* \notin \Delta\}$ is a 2-valued stable model, while $P^* \cup \Delta$ satisfies I^* if we take $\Delta = \{a^* \mid a \notin M\}$ for a 2-valued stable model M . Δ is expected as an abductive explanation for a query. But the following example shows that there is no Δ such that $P^* \cup \Delta$ satisfies I^* .

Example 3.2 Assume a general logic program P :

$$\left\{ \begin{array}{l} t \leftarrow \neg r, \\ r \leftarrow \neg q \text{ s}, \\ q \leftarrow p, \\ p \leftarrow \neg q \end{array} \right\}.$$

There is just one 3-valued stable model $\langle \{t\}, \{s, r\} \rangle$ of P , but no 2-valued stable model. Hence there is no Δ such that $P^* \cup \Delta$ satisfies I^* .

Since there exists a 3-valued stable model for any general logic program [17], we are hereby motivated to consider an adequate constraint in correlation with a 3-valued stable model. As well, we devise an abductive proof procedure which should be in a more close relation with the newly considered constraint, though E-K procedure is sound with respect to a 3-valued stable model [4].

3.2 3-Valued Integrity Constraint

We present an integrity constraint which an abductive explanation is denoted by, and which is in correlation with a 3-valued stable model.

Given a general logic program P , we define $B_{P^*} = B_P \cup (B_P)^*$. For an atom set S , let $S^u = \{a \leftrightarrow u \mid a \in S\}$.

Definition 3.3 An amended abduction framework is a triplet $\langle P^*, Ab^*, I_t^* \rangle$, where

(1) P^* is the same as the set in the former framework.

(2) Ab^* is the same as the set in the former framework.

(3) I_t^* is the set of all integrity constraints of the form:

$$\begin{aligned} & \forall \bar{x} \neg [p(\bar{x}) \wedge (p^*(\bar{x}) \leftarrow u)] \text{ and } \forall \bar{x} \neg [(p(\bar{x}) \leftarrow u) \wedge p^*(\bar{x})] \\ & \text{and} \\ & \forall \bar{x} [p(\bar{x}) \vee p^*(\bar{x}) \vee ((p(\bar{x}) \leftarrow u) \wedge (p^*(\bar{x}) \leftarrow u))]. \end{aligned}$$

For $\Gamma, \Delta \subseteq Ab^*$, $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* iff for any $s \in B_P$

- (1) $P^* \cup \Gamma^u \cup \Delta \not\models_3 s \wedge (s^* \leftarrow u)$, and
- (2) $P^* \cup \Gamma^u \cup \Delta \not\models_3 (s \leftarrow u) \wedge s^*$, and
- (3) $P^* \cup \Gamma^u \cup \Delta \models_3 s$ or $P^* \cup \Gamma^u \cup \Delta \models_3 s^*$ or $P^* \cup \Gamma^u \cup \Delta \models_3 (s \leftarrow u) \wedge (s^* \leftarrow u)$.

Γ is expected as an adjustment, while Δ as an abductive explanation. It is easy to see the following lemma.

Lemma 3.4 *Assume an abduction framework $\langle P^*, Ab^*, I_t^* \rangle$ and $\Gamma, \Delta \subseteq Ab^*$. If $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* , then $\Gamma \cap \Delta = \emptyset$.*

Proof Assume that $\Gamma \cap \Delta \neq \emptyset$. Then there is no 3-valued Herbrand model of $P^* \cup \Gamma^u \cup \Delta$, and any formula is a logical consequence of $P^* \cup \Gamma^u \cup \Delta$, which contradicts the constraint I_t^* . Hence $\Gamma \cap \Delta = \emptyset$. q.e.d.

On the other hand, if $\Gamma \cap \Delta = \emptyset$, then there is a 3-valued Herbrand model of $P^* \cup \Gamma^u \cup \Delta$:

On the assumption that $\Gamma \cap \Delta = \emptyset$, there is no case that $a^* \in \Gamma$ and $a^* \in \Delta$. Therefore there is always a 3-valued Herbrand model of $\Gamma^u \cup \Delta$. There is no clause of P^* , whose head has a predicate with $*$. Hence there is a 3-valued Herbrand model which satisfies P^* and $\Gamma \cap \Delta$.

3.3 Consistency of Constraints

In this section, we present several lemmas, whose proofs are shown in Appendix, as regards constraints. The main purpose is to express any logical consequence of $P^* \cup \Gamma^u \cup \Delta$ by means of the memberships in the sets $\Gamma \cup \Delta$, Δ , $S_P(\Delta^+)$ and $S_P(\Gamma^+ \cup \Delta^+)$.

Observing P^{Δ^+} in defining $S_P(\Delta^+)$, we see that P^{Δ^+} is regarded as equivalent to $P^* \cup \Delta$ in the following sense.

Lemma 3.5 *Assume an abduction framework $\langle P^*, Ab^*, I_t^* \rangle$ and $\Delta \subseteq Ab^*$. Then*

$$S_P(\Delta^+) = \{a \in B_P \mid P^* \cup \Delta \models a\}.$$

Proof See the appendix.

We can see that any member of Δ is a logical consequence of $P^* \cup \Delta$ and vice versa:

Lemma 3.6 *Assume an abduction framework $\langle P^*, Ab^*, I_t^* \rangle$ and $\Delta \subseteq Ab^*$. $a^* \in \Delta$ iff $P^* \cup \Delta \models a^*$.*

Proof See the appendix.

Noting that $P^* \cup \Delta$ is a set of definite clauses (clauses involving no negative literals in their bodies), we have:

Lemma 3.7 *Assume an abduction framework $\langle P^*, Ab^*, I_t^* \rangle$ and $\Delta \subseteq Ab^*$. For $l \in B_{P^*}$, $P^* \cup \Delta \models l$ iff $P^* \cup \Delta \models_3 l$.*

Proof See the appendix.

Now we investigate the relation between logical consequences of $P^* \cup \Delta$ and $P^* \cup \Gamma^u \cup \Delta$.

Lemma 3.8 Assume an abduction framework $\langle P^*, Ab^*, I_t^* \rangle$ and $\Gamma, \Delta \subseteq Ab^*$ such that $\Gamma \cap \Delta = \emptyset$. Then

(1) A 3-valued Herbrand interpretation $M = \langle S_P(\Delta^+) \cup \Delta, \emptyset \rangle$ is a model of $P^* \cup \Gamma^u \cup \Delta$.

(2) For $l \in B_{P^*}$, $P^* \cup \Delta \models_3 l$ iff $P^* \cup \Gamma^u \cup \Delta \models_3 l$.

Proof See the appendix.

By means of Lemmas 3.5, 3.6, 3.7 and 3.8, we have:

Corollary 3.9 Assume an abduction framework $\langle P^*, Ab^*, I_t^* \rangle$ and $\Gamma, \Delta \subseteq Ab^*$ such that $\Gamma \cap \Delta = \emptyset$. For $l \in B_{P^*}$,

$$\begin{aligned} & P^* \cup \Delta \models l \\ \Leftrightarrow & P^* \cup \Delta \models_3 l \\ \Leftrightarrow & P^* \cup \Gamma^u \cup \Delta \models_3 l \\ \Leftrightarrow & [(l \in B_P \Rightarrow l \in S_P(\Delta^+)) \wedge (l \in (B_P)^* \Rightarrow l \in \Delta)] \end{aligned}$$

What follows, the relations between models of $P^* \cup \Gamma^u \cup \Delta$ and $P^* \cup \Delta$ are shown.

Lemma 3.10 Assume an abduction framework $\langle P^*, Ab^*, I_t^* \rangle$ and $\Gamma, \Delta \subseteq Ab^*$ such that $\Gamma \cap \Delta = \emptyset$.

(1) If $\langle IT, IF \rangle$ is a 3-valued Herbrand model of $P^* \cup \Gamma^u \cup \Delta$, then $B_{P^*} - IF$ is a 2-valued Herbrand model of $P^* \cup \Gamma \cup \Delta$.

(2) If I is a 2-valued Herbrand model of $P^* \cup \Gamma \cup \Delta$, then $\langle I - \Gamma, B_{P^*} - I \rangle$ is a 3-valued Herbrand model of $P^* \cup \Gamma^u \cup \Delta$.

(3) For any $l \in B_{P^*}$, $P^* \cup \Gamma^u \cup \Delta \models_3 l \leftarrow u$ iff $P^* \cup \Gamma \cup \Delta \models l$.

Proof See the appendix.

By Lemmas 3.5, 3.6 and 3.10, we have:

Corollary 3.11 Assume an abduction framework $\langle P^*, Ab^*, I_t^* \rangle$ and $\Gamma, \Delta \subseteq Ab^*$ such that $\Gamma \cap \Delta = \emptyset$. For $l \in B_{P^*}$,

$$\begin{aligned} & P^* \cup \Gamma^u \cup \Delta \models_3 l \leftarrow u \\ \Leftrightarrow & P^* \cup \Gamma \cup \Delta \models l \\ \Leftrightarrow & [(l \in B_P \Rightarrow l \in S_P(\Gamma^+ \cup \Delta^+)) \wedge (l \in (B_P)^* \Rightarrow l \in \Gamma \cup \Delta)] \end{aligned}$$

As regards the logical consequence of $P^* \cup \Gamma^u \cup \Delta$, we have:

Lemma 3.12 Assume an abduction framework $\langle P^*, Ab^*, I_t^* \rangle$ and $\Gamma, \Delta \subseteq Ab^*$ such that $\Gamma \cap \Delta = \emptyset$.

(1) For $s^* \in Ab^*$, $s^* \in \Delta$ iff $P^* \cup \Gamma^u \cup \Delta \models_3 s^*$.

(2) For $s^* \in Ab^*$, $s^* \in \Gamma$ iff $P^* \cup \Gamma^u \cup \Delta \models_3 s^* \leftrightarrow u$.

(3) For $s^* \in Ab^*$, $s^* \notin \Gamma \cup \Delta$ iff $P^* \cup \Gamma^u \cup \Delta \not\models_3 s^* \leftarrow u$.

Proof See the appendix.

We now see the consistency of the constraint when $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* . Let

$$\begin{aligned} \mathbf{T} &= \{l \in B_{P^*} \mid P^* \cup \Gamma^u \cup \Delta \models_3 l\}, \\ \mathbf{F} &= \{l \in B_{P^*} \mid P^* \cup \Gamma^u \cup \Delta \not\models_3 l \leftarrow \mathbf{u}\}, \text{ and} \\ \mathbf{U} &= B_{P^*} - (\mathbf{T} \cup \mathbf{F}). \end{aligned}$$

Lemma 3.13 Assume an abduction framework $\langle P^*, Ab^*, I_t^* \rangle$ and $\Gamma, \Delta \subseteq Ab^*$ such that $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* . For any $a \in B_P$,

- (1) $a \in \mathbf{T} \Leftrightarrow a^* \in \mathbf{F}$.
- (2) $a \in \mathbf{F} \Leftrightarrow a^* \in \mathbf{T}$.
- (3) $a \in \mathbf{U} \Leftrightarrow a^* \in \mathbf{U}$.

Proof See the appendix.

Lemma 3.14 Assume an abduction framework $\langle P^*, Ab^*, I_t^* \rangle$ and $\Gamma, \Delta \subseteq Ab^*$ such that $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* . Then for any $a \in B_P$,

- (1) $a \in \mathbf{T} \Leftrightarrow a^* \in \mathbf{F} \Leftrightarrow a \in S_P(\Delta^+) \Leftrightarrow a^* \notin \Gamma \cup \Delta$.
- (2) $a \in \mathbf{F} \Leftrightarrow a^* \in \mathbf{T} \Leftrightarrow a \notin S_P(\Gamma^+ \cup \Delta^+) \Leftrightarrow a^* \in \Delta$.
- (3) $a \in \mathbf{U} \Leftrightarrow a^* \in \mathbf{U} \Leftrightarrow a \in S_P(\Gamma^+ \cup \Delta^+) \wedge a \notin S_P(\Delta^+) \Leftrightarrow a^* \in \Gamma$.

Proof See the appendix.

3.4 Relation between Constraint and Stable Model

We have the relations between the constraint I_t^* and the 3-valued stable model by making use of the alternation fixpoint semantics [1, 9]. That is, whether $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* is equivalent to that $\langle S_P(\Delta^+), \Delta^+ \rangle$ is a 3-valued stable model such that $\Gamma^+ = \overline{\Delta^+ \cup S_P(\Delta^+)}$. We see this relation in what follows.

Lemma 3.15 Assume $\Gamma, \Delta \subseteq Ab^*$. If $\Delta^+ \cap S_P(\Delta^+) = \emptyset$ and $\Gamma^+ = \overline{\Delta^+ \cup S_P(\Delta^+)}$, then $\Gamma^+ \cup \Delta^+ = S_P(\Delta^+)$.

Proof

$$\begin{aligned} \Gamma^+ \cup \Delta^+ &= \overline{\Delta^+ \cup S_P(\Delta^+)} \cup \Delta^+ \\ &= S_P(\Delta^+) \quad (\text{by } \Delta^+ \cap S_P(\Delta^+) = \emptyset) \end{aligned}$$

q.e.d.

Theorem 3.16 Assume an abduction framework $\langle P^*, Ab^*, I_t^* \rangle$. $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* for $\Gamma, \Delta \subseteq Ab^*$ iff

- (1) $\Delta^+ \cap S_P(\Delta^+) = \emptyset$,
- (2) $\Gamma^+ = \overline{\Delta^+ \cup S_P(\Delta^+)}$, and
- (3) $\Delta^+ = A_P(\Delta^+)$.

Proof (Only if part) (1)

$$\begin{aligned} a^* \in \Delta &\Leftrightarrow a \notin S_P(\Gamma^+ \cup \Delta^+) \text{ (by Lemma 3.14)} \\ &\Rightarrow a \notin S_P(\Delta^+). \text{ (by monotonicity of } S_P) \end{aligned}$$

(2)

$$\begin{aligned}
 a^* \in \Gamma &\Leftrightarrow a \in S_P(\Gamma^+ \cup \Delta^+) \wedge a \notin S_P(\Delta^+) \\
 &\quad \text{(by Lemma 3.14)} \\
 &\Leftrightarrow a \notin \Delta^+ \wedge a \notin S_P(\Delta^+) \text{ (by Lemma 3.14)} \\
 &\Leftrightarrow a \notin \Delta^+ \cup S_P(\Delta^+)
 \end{aligned}$$

(3)

$$\begin{aligned}
 a^* \in \Delta &\Leftrightarrow a \notin S_P(\Gamma^+ \cup \Delta^+) \text{ (by Lemma 3.14)} \\
 &\Leftrightarrow a \notin S_P(\overline{S_P(\Delta^+)}) \text{ (by the lemma 3.15} \\
 &\quad \text{as well as (1) and (2))} \\
 &\Leftrightarrow a \in \overline{S_P(\overline{S_P(\Delta^+)})} = A_P(\Delta^+).
 \end{aligned}$$

(If part) (i) Assume that $P^* \cup \Gamma^u \cup \Delta \models_3 s \wedge (s^* \leftarrow u)$ for some $s \in B_P$, on the contrary to the constraint. By Lemma 3.14, $s \in S_P(\Delta^+)$ and $s^* \in \Gamma \cup \Delta$. If $s^* \in \Gamma$, then

$$s^* \in \Gamma \Leftrightarrow s \in \overline{\Delta^+ \cup S_P(\Delta^+)} \Leftrightarrow s \notin \Delta^+ \cup S_P(\Delta^+),$$

which contradicts to that $s \in S_P(\Delta^+)$. If $s^* \in \Delta$, then $s \in \Delta^+ \cap S_P(\Delta^+)$, which contradicts that $\Delta^+ \cap S_P(\Delta^+) = \emptyset$. Hence $P^* \cup \Gamma^u \cup \Delta \not\models_3 s \wedge (s^* \leftarrow u)$ for any s .

(ii) Assume that $P^* \cup \Gamma^u \cup \Delta \models_3 (s \leftarrow u) \wedge s^*$, on the contrary to the constraint. By Lemma 3.14, $s \in S_P(\Gamma^+ \cup \Delta^+)$ and $s^* \in \Delta$. By Lemma 3.14,

$$s \in S_P(\Gamma^+ \cup \Delta^+) = S_P(\overline{S_P(\Delta^+)}) \Leftrightarrow s \notin \overline{S_P(\overline{S_P(\Delta^+)})} = A_P(\Delta^+) = \Delta^+,$$

which contradicts that $s \in \Delta^+$. Hence $P^* \cup \Gamma^u \cup \Delta \not\models_3 (s \leftarrow u) \wedge s^*$ for any s .

(iii) Assume that $P^* \cup \Gamma^u \cup \Delta \not\models_3 s$, $P^* \cup \Gamma^u \cup \Delta \not\models_3 s^*$ and $P^* \cup \Gamma^u \cup \Delta \not\models_3 (s \leftarrow u) \wedge (s^* \leftarrow u)$ for some s , on the contrary to the constraint. It follows from Lemma 3.14 that $s \notin S_P(\Delta^+)$ and $s \notin \Delta^+$. As well,

$$P^* \cup \Gamma^u \cup \Delta \not\models_3 s \leftarrow u \text{ or } P^* \cup \Gamma^u \cup \Delta \not\models_3 s^* \leftarrow u.$$

If $P^* \cup \Gamma^u \cup \Delta \not\models_3 s \leftarrow u$, then

$$\begin{aligned}
 P^* \cup \Gamma^u \cup \Delta \not\models_3 s \leftarrow u &\Leftrightarrow s \notin S_P(\Gamma^+ \cup \Delta^+) && \text{(by Lemma 3.14)} \\
 &\Leftrightarrow s \notin \overline{S_P(\overline{S_P(\Delta^+)})} && \text{(by Lemma 3.15)} \\
 &\Leftrightarrow s \in \overline{S_P(\overline{S_P(\Delta^+)})} = A_P(\Delta^+) = \Delta^+,
 \end{aligned}$$

which contradicts that $s \notin \Delta^+$. If $P^* \cup \Gamma^u \cup \Delta \not\models_3 s^* \leftarrow u$, then

$$\begin{aligned}
 P^* \cup \Gamma^u \cup \Delta \not\models_3 s^* \leftarrow u &\Leftrightarrow s \notin \Gamma^+ \cup \Delta^+ && \text{(by Lemma 3.14)} \\
 &\Leftrightarrow s \notin \overline{S_P(\Delta^+)} && \text{(by Lemma 3.15)} \\
 &\Leftrightarrow s \in S_P(\Delta^+),
 \end{aligned}$$

which contradicts that $s \notin S_P(\Delta^+)$. Hence for any s

$$P^* \cup \Gamma^u \cup \Delta \models_3 s, P^* \cup \Gamma^u \cup \Delta \models_3 s^*, \text{ or } P^* \cup \Gamma^u \cup \Delta \models_3 (s \leftarrow u) \wedge (s^* \leftarrow u).$$

This concludes the proof.

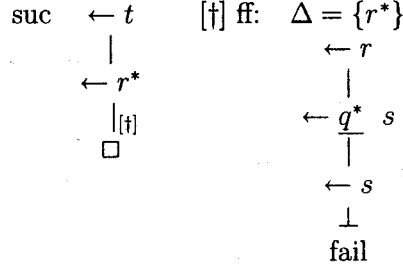
q.e.d.

By Theorems 2.8 and 3.16, we have:

Theorem 3.17 *On the assumption that $\langle P^*, Ab^*, I_t^* \rangle$ is an abduction framework and $\Gamma, \Delta \subseteq Ab^*$, $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* iff $\langle S_P(\Delta^+), \Delta^+ \rangle$ is a 3-valued stable model and $\Gamma^+ = \overline{\Delta^+ \cup S_P(\Delta^+)}$.*

4 Augmented Abductive Proof Procedure

For a general logic program P as in Example 3.2, E-K procedure demonstrates the derivations with P^* obtained from P :



where suc stands for a succeeding derivation and ff for a finitely failing derivation. Because there is no succeeding derivation from $\leftarrow q$, $\leftarrow q^*$ is removed from the above ff. However, we present an augmented procedure in which such a removal is replaced by an adjustment or an explanation. This replacement is regarded as a reflection of some 3-valued stable model, on the basis of the third truth value, that is, **u**.

4.1 Abductive Proof Procedure

The augmented abductive proof procedure consists of three derivations: an abductive succeeding derivation (suc, for short), an abductive finitely failing derivation (ff, for short) and an abductive adjusting derivation (adj, for short).

A normal goal of the form $\leftarrow L_1 \dots L_n$ is called an abductive goal, where each L_i is in B_{P^*} . By $mgu(A, A')$ we mean one of most general unifiers of A and A' for given atoms A and A' . For a substitution θ , $\theta|_G$ stands for a substitution which can operate on just variables involved in G . On the assumption of an abduction framework $\langle P^*, Ab^*, I_t^* \rangle$, an abductive proof procedure is recursively defined by means of three derivations, when a safe rule R of selection of literals for abductive goals is given. Note that no nonground literal is selected by R . An expression $E\theta$ obtained by applying a substitution θ to an expression E is explained in [14]. Also see it for other terminologies as regards logic programming.

(1) An abductive succeeding derivation (suc):

An abductive succeeding derivation from G^* of rank r and length h is a sequence of quadruplets of an abductive goal, a substitution, an adjustment and an abductive explanation

$$(G_0^*, \theta_0, \Gamma_0, \Delta_0), \dots, (G_h^*, \theta_h, \Gamma_h, \Delta_h),$$

where $G_0^* = G^*$, $G_h^* = \square$ and the sequence is organized by the following rules. When $\theta_0 = \varepsilon$, the above abductive succeeding derivation is denoted by $(G_0^*, \Gamma_0, \Delta_0) \rightsquigarrow_{\text{suc}}^{\theta'} (\Gamma_h, \Delta_h)$ for $\theta' = \theta_h|_{G_0^*}$, or $(G_0^*, \Gamma_0, \Delta_0) \rightsquigarrow_{\text{suc}} (\theta_h, \Gamma_h, \Delta_h)$, without indicating the rank.

(Rules) Let $G_k^* \equiv \leftarrow L_1 \dots L_n$, where L_i is selected by the safe rule R . $(G_{k+1}^*, \theta_{k+1}, \Gamma_{k+1}, \Delta_{k+1})$ is obtained from $(G_k^*, \theta_k, \Gamma_k, \Delta_k)$ by:

(suc1) In case that there is $A' \leftarrow l_1 \dots l_m \in P^*$ such that $L_i = A$ and $\theta = mgu(A, A')$,

$$\begin{aligned}
 G_{k+1}^* &\equiv \leftarrow (L_1 \dots L_{i-1} l_1 \dots l_m L_{i+1} \dots L_n) \theta, \\
 \theta_{k+1} &= \theta_k \theta, \Gamma_{k+1} = \Gamma_k, \Delta_{k+1} = \Delta_k.
 \end{aligned}$$

(suc2) In case that $L_i = A^*$ (a ground atom) and $A^* \in \Delta_k$,

$$\begin{aligned} G_{k+1}^* &\equiv \leftarrow L_1 \dots L_{i-1} L_{i+1} \dots L_n, \\ \theta_{k+1} &= \theta_k, \Gamma_{k+1} = \Gamma_k, \Delta_{k+1} = \Delta_k. \end{aligned}$$

(suc3) In case that $L_i = A^*$ (a ground atom), $A^* \notin \Gamma_k \cup \Delta_k$ and there is an abductive finitely failing derivation of rank r' ($< r$), that is,

$$(\{\leftarrow A\}, \Gamma_k, \Delta_k \cup \{A^*\}) \rightsquigarrow_{ff} (\Gamma', \Delta'),$$

$$\begin{aligned} G_{k+1}^* &\equiv \leftarrow L_1 \dots L_{i-1} L_{i+1} \dots L_n, \\ \theta_{k+1} &= \theta_k, \Gamma_{k+1} = \Gamma', \Delta_{k+1} = \Delta'. \end{aligned}$$

(2) An abductive finitely failing derivation (ff):

For a set F of abductive goals, an abductive finitely failing derivation of rank r and length h is a sequence of triplets of a set of abductive goals, an adjustment and an abductive explanation

$$(F_0, \Gamma_0, \Delta_0), \dots, (F_h, \Gamma_h, \Delta_h),$$

where $F_0 = F$, $F_h = \emptyset$, $\square \notin F_k$ for each k , and the sequence is organized by the following rules. The above abductive finitely failing derivation is denoted by $(F_0, \Gamma_0, \Delta_0) \rightsquigarrow_{ff} (\Gamma_h, \Delta_h)$, without indicating the rank.

(Rules) Assume that $F_k = F'_k \cup \{\leftarrow L_1 \dots L_n\}$, where L_i is selected by the safe rule R in $\leftarrow L_1 \dots L_n$.

(ff1) In case that $L_i = A$,

$$\begin{aligned} F_{k+1} &= F'_k \cup \{G_1^*, \dots, G_m^*\}, \\ \Gamma_{k+1} &= \Gamma_k, \Delta_{k+1} = \Delta_k, \end{aligned}$$

where

$$G_j^* \equiv \leftarrow (L_1 \dots L_{i-1} l_1^j \dots l_{q_j}^j L_{i+1} \dots L_n) \theta^j$$

is a derived abductive goal for $A^j \leftarrow l_1^j \dots l_{q_j}^j \in P^*$ and $\theta^j = \text{mgu}(A, A^j)$.

(ff2) In case that $L_i = A^*$ (a ground atom) and $A^* \in \Gamma_k \cup \Delta_k$,

$$\begin{aligned} F_{k+1} &= F'_k \cup \{\leftarrow L_1 \dots L_{i-1} L_{i+1} \dots L_n\}, \\ \Gamma_{k+1} &= \Gamma_k, \Delta_{k+1} = \Delta_k. \end{aligned}$$

(ff3) In case that $L_i = A^*$ (a ground atom) and $A^* \notin \Gamma_k \cup \Delta_k$,

(ff3-1) if there is an abductive succeeding derivation of rank r' ($< r$),

$$(\leftarrow A, \Gamma_k, \Delta_k) \rightsquigarrow_{suc}^\epsilon (\Gamma', \Delta'),$$

then

$$F_{k+1} = F'_k, \Gamma_{k+1} = \Gamma', \Delta_{k+1} = \Delta',$$

(ff3-2) if there is an abductive finitely failing derivation of rank r' ($< r$), that is,

$$(\{\leftarrow A\}, \Gamma_k, \Delta_k \cup \{A^*\}) \rightsquigarrow_{ff} (\Gamma', \Delta'),$$

then

$$\begin{aligned} F_{k+1} &= F'_k \cup \{\leftarrow L_1 \dots L_{i-1} L_{i+1} \dots L_n\}, \\ \Gamma_{k+1} &= \Gamma', \Delta_k = \Delta', \end{aligned}$$

(ff3-3) if there is an abductive adjusting derivation of rank r' ($< r$), that is,

$$(\{(t, \leftarrow A)\}, \Gamma_k \cup \{A^*\}, \Delta_k) \rightsquigarrow_{adj} (\Gamma', \Delta'),$$

then

$$\begin{aligned} F_{k+1} &= F'_k \cup \{\leftarrow L_1 \dots L_{i-1} L_{i+1} \dots L_n\}, \\ \Gamma_{k+1} &= \Gamma', \Delta_{k+1} = \Delta', \end{aligned}$$

(ff3-4) otherwise,

$$\begin{aligned} F_{k+1} &= F'_k \cup \{\leftarrow L_1 \dots L_{i-1} L_{i+1} \dots L_n\}, \\ \Gamma_{k+1} &= \Gamma_k, \Delta_{k+1} = \Delta_k. \end{aligned}$$

(3) An abductive adjusting derivation (adj):

An abductive adjusting derivation from C of rank r and of length h is a sequence

$$(C_0, \Gamma_0, \Delta_0), \dots, (C_h, \Gamma_h, \Delta_h),$$

where $C_0 = C$, each C_k has the form (V, G^*) for $V = \mathbf{t}$ or $\mathbf{u}$, $(\mathbf{t}, \Box) \notin C_k$ ($0 \leq k \leq h$), $C_h = \{(\mathbf{u}, \Box)\}$, and the sequence is organized by the following rules. The above abductive adjusting derivation is denoted by $(C_0, \Gamma_0, \Delta_0) \rightsquigarrow_{adj} (\Gamma_h, \Delta_h)$, without indicating the rank.

(Rules) Let $C_k = C'_k \cup \{(V, \leftarrow L_1 \dots L_n)\}$, where L_i is selected by the safe rule R .

(adj1) In case that $L_i = A$,

$$\begin{aligned} C_{k+1} &= C'_k \cup \{(V, G_1^*), \dots, (V, G_m^*)\}, \\ \Gamma_{k+1} &= \Gamma_k, \Delta_{k+1} = \Delta_k, \end{aligned}$$

where

$$G_j^* \equiv \leftarrow (L_1 \dots L_{i-1} l_1^j \dots l_{q_j}^j L_{i+1} \dots L_n) \theta^j$$

is a derived abductive goal for $A^j \leftarrow l_1^j \dots l_{q_j}^j \in P^*$ and $\theta^j = mgu(A, A^j)$.

(adj2) In case that $L_i = A^*$ (a ground atom), $A^* \in \Gamma_k$ and $A^* \notin \Delta_k$,

$$\begin{aligned} C_{k+1} &= C'_k \cup \{(\mathbf{u}, \leftarrow L_1 \dots L_{i-1} L_{i+1} \dots L_n)\}, \\ \Gamma_{k+1} &= \Gamma_k, \Delta_{k+1} = \Delta_k. \end{aligned}$$

(adj3) In case that $L_i = A^*$ (a ground atom) and $A^* \in \Delta_k$,

$$\begin{aligned} C_{k+1} &= C'_k \cup \{(V, \leftarrow L_1 \dots L_{i-1} L_{i+1} \dots L_n)\}, \\ \Gamma_{k+1} &= \Gamma_k, \Delta_{k+1} = \Delta_k. \end{aligned}$$

(adj4) In case that $L_i = A^*$ (a ground atom) and $A^* \notin \Gamma_k \cup \Delta_k$,

(adj4-1) if there is an abductive adjusting derivation of rank r' ($< r$), that is,

$$(\{(t, \leftarrow A)\}, \Gamma_k \cup \{A^*\}, \Delta_k) \rightsquigarrow_{adj} (\Gamma', \Delta'),$$

then

$$\begin{aligned} C_{k+1} &= C'_k \cup \{(\mathbf{u}, \leftarrow L_1 \dots L_{i-1} L_{i+1} \dots L_n)\}, \\ \Gamma_{k+1} &= \Gamma', \Delta_{k+1} = \Delta', \end{aligned}$$

(adj4-2) if there is an abductive succeeding derivation of rank r' ($< r$), that is,

$$(\leftarrow A, \Gamma_k, \Delta_k) \rightsquigarrow_{suc}^\varepsilon (\Gamma', \Delta'),$$

then

$$C_{k+1} = C'_k, \Gamma_{k+1} = \Gamma', \Delta_{k+1} = \Delta',$$

(adj4-3) if there is an abductive finitely failing derivation of rank r' ($< r$), that is,

$$(\{\leftarrow A\}, \Gamma_k, \Delta_k \cup \{A^*\}) \rightsquigarrow_{ff} (\Gamma', \Delta'),$$

then

$$\begin{aligned} C_{k+1} &= C'_k \cup \{(V, \leftarrow L_1 \dots L_{i-1} L_{i+1} \dots L_n)\}, \\ \Gamma_{k+1} &= \Gamma', \Delta_{k+1} = \Delta'. \end{aligned}$$

Example 4.1 Assume a general logic program P as in Example 3.2. With P^* induced from P , we have the following abductive adjusting derivation [‡] for $\leftarrow q^*$. For an abductive goal $\leftarrow r$, we have the following finitely failing derivation: $(\{\leftarrow r\}, \emptyset, \emptyset) \rightsquigarrow_{ff} (\{q^*\}, \emptyset)$. As well, $(\leftarrow t, \emptyset, \emptyset) \rightsquigarrow_{suc}^\varepsilon (\{q^*\}, \{r^*\})$. On the other hand, for an abductive goal $\leftarrow q$, we have the following abductive adjusting derivation: $(\{\leftarrow t, \leftarrow q\}, \emptyset, \emptyset) \rightsquigarrow_{adj} (\{q^*\}, \emptyset)$.

[‡] adj:	$\Gamma = \{q^*\}$	ff:	$\Delta = \emptyset$	adj:	$\Gamma = \emptyset$
	$(\mathbf{t}, \leftarrow q)$		$\leftarrow r$		$(\mathbf{t}, \leftarrow q)$
	$(\mathbf{t}, \leftarrow p)$		$\leftarrow q^* \ s$		$(\mathbf{t}, \leftarrow p)$
			[‡]		
	$(\mathbf{t}, \leftarrow q^*)$		$\leftarrow s$		$(\mathbf{u}, \leftarrow q^*)$
			$\perp$		[‡]
	$(\mathbf{u}, \square)$		fail		$(\mathbf{u}, \square)$

To stand for any derivation suc, ff or adj, we adopt the following notation:

Definition 4.2 By $(\Gamma_0, \Delta_0) \rightsquigarrow_{any} (\Gamma, \Delta)$, we mean $(G^*, \Gamma_0, \Delta_0) \rightsquigarrow_{suc}^\theta (\Gamma, \Delta)$ for some G^* and θ , $(F, \Gamma_0, \Delta_0) \rightsquigarrow_{ff} (\Gamma, \Delta)$ for some F , or $(C, \Gamma_0, \Delta_0) \rightsquigarrow_{adj} (\Gamma, \Delta)$ for some C .

By the definitions of suc, ff, and adj, we have:

- If $(\{\leftarrow A\}, \Gamma_0, \Delta_0) \rightsquigarrow_{ff} (\Gamma, \Delta)$, then $A^* \in \Delta_0$.
- If $(\{\leftarrow t, \leftarrow A\}, \Gamma_0, \Delta_0) \rightsquigarrow_{adj} (\Gamma, \Delta)$, then $A^* \in \Gamma_0$.

We follow [4, 6] for the notations to show the recursive relations between derivations.

Definition 4.3 If a subderivation $\gamma' : (\Gamma_a, \Delta_a) \rightsquigarrow_{any} (\Gamma'_a, \Delta'_a)$ appears in a derivation $\gamma : (\Gamma_0, \Delta_0) \rightsquigarrow_{any} (\Gamma, \Delta)$, then it is denoted by $\gamma > \gamma'$. $\gg$ is the transitive closure of $>$.

From now on, $S \in \gamma$ means that a configuration S appears in a derivation γ . For example, $(G^*, \theta, \Gamma, \Delta) \in \beta$ means that $(G^*, \theta, \Gamma, \Delta)$ appears in β .

4.2 Soundness of Abductive Proof Procedure

Before seeing the soundness of the presented abductive proof procedure, we need several lemmas. Assume an abduction framework $\langle P^*, Ab^*, I_t^* \rangle$.

1. Succeeding, finitely failing and adjusting derivations

Intuitively speaking, the following lemmas state that if there is a succeeding derivation from an abductive goal with initial sets then any finitely failing derivation cannot be expected with the same set, nor an adjusting derivation. On the other hand, any finitely failing derivation, which is a subderivation of any derivation with Γ and Δ as results, cannot permit any succeeding derivation, with the super sets of sets Γ and Δ as initial sets, for the abductive goal which occurs in the finitely failing derivations.

If there is an abductive succeeding derivation with results Γ and Δ , then each atom involved in the abductive goal is a logical consequence of $P^* \cup \Gamma^u \cup \Delta$. This is stated by:

Lemma 4.4 Assume that $(\leftarrow L_1 \dots L_n, \Gamma_0, \Delta_0) \rightsquigarrow_{suc} (\theta, \Gamma, \Delta)$. Then

$$\forall i : [(1 \leq i \leq n) \Rightarrow [(L_i \theta \varphi = a \in B_P \Rightarrow a \in S_P(\Delta^+)) \wedge (L_i \theta \varphi = a^* \in (B_P)^* \Rightarrow a^* \in \Delta)]].$$

Proof See the appendix.

When Γ and Δ are obtained by an abductive succeeding derivation, there is neither an abductive finitely failing derivation, nor an abductive adjusting derivation, if Γ' and Δ' are initial sets such that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. On the other hand, there is an abductive succeeding derivation with Γ' and Δ' as initial sets such that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. This is formally given by:

Lemma 4.5 Assume that $(G^*, \Gamma_0, \Delta_0) \rightsquigarrow_{suc} (\theta, \Gamma, \Delta)$. Then

$$\begin{aligned} \forall \Gamma', \Delta', \Gamma'', \Delta'' : [(\Gamma \subseteq \Gamma') \wedge (\Delta \subseteq \Delta') \Rightarrow \\ (G^*, \Gamma', \Delta') \rightsquigarrow_{suc}^{\theta} (\Gamma', \Delta') \\ \wedge (\{G^*\}, \Gamma', \Delta') \not\rightsquigarrow_{ff} (\Gamma'', \Delta'') \\ \wedge (\{t, G^*\}, \Gamma', \Delta') \not\rightsquigarrow_{adj} (\Gamma'', \Delta'')] \end{aligned}$$

Proof See the appendix.

If there is an abductive finitely failing derivation in some derivation with Γ and Δ as results, then there is no abductive succeeding derivation with the initial sets Γ' and Δ' such that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. It is shown below:

Lemma 4.6 Assume $\gamma : (\Gamma_0, \Delta_0) \rightsquigarrow_{any} (\Gamma, \Delta)$ and $\mu : (\{ \leftarrow B \}, \Gamma_a, \Delta_a) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$ such that $\gamma \gg \mu$. Let

$$\begin{aligned} & \Delta'' \\ = & \Delta \cup \{s^* \mid \exists \Gamma_c, \Delta_c, \Gamma'_c, \Delta'_c : [(\Gamma \subseteq \Gamma_c) \wedge (\Delta \subseteq \Delta_c) \wedge (\{ \leftarrow s \}, \Gamma_c, \Delta_c) \rightsquigarrow_{ff} (\Gamma'_c, \Delta'_c)]\}. \end{aligned}$$

If $(\{G_1^*, \dots, G_l^*\}, \Gamma_b, \Delta_b) \in \mu$, then

$$\begin{aligned} \forall i, \Gamma', \Delta', \theta_z, \Gamma_z, \Delta_z : [(1 \leq i \leq l) \Rightarrow [(\Gamma \subseteq \Gamma') \wedge (\Delta \subseteq \Delta' \subseteq \Delta'') \\ \Rightarrow (G_i^*, \Gamma', \Delta') \not\rightsquigarrow_{suc}^{\theta_z} (\Gamma_z, \Delta_z)]]]. \end{aligned}$$

Proof See the appendix.

If there is an adjusting derivation with Γ and Δ as results, occurring in some derivation, then there is also an adjusting derivation starting with Γ and Δ for some abductive goal occurring in the adjusting derivation. This is given formally:

Lemma 4.7 Assume that $\gamma : (\Gamma_0, \Delta_0) \rightsquigarrow_{any} (\Gamma, \Delta)$ and $\nu : (\{(t, \leftarrow B)\}, \Gamma_a, \Delta_a) \rightsquigarrow_{adj} (\Gamma'_a, \Delta'_a)$ such that $\gamma \gg \nu$. Then

$$(C, \Gamma_b, \Delta_b) \in \nu \Rightarrow (C, \Gamma, \Delta) \rightsquigarrow_{adj} (\Gamma, \Delta).$$

Proof See the appendix.

2. Adjustments and abducibles obtained by succeeding derivations

If there is an abductive succeeding derivation with $\leftarrow a$ as an abductive goal and with Γ_0, Δ_0 as the initial sets such that $a^* \notin \Gamma_0^*$ and $a^* \notin \Delta_0^*$, then $a^* \notin \Gamma$ and $a^* \notin \Delta$ for Γ and Δ obtained by the derivation. In a more formal description, we have:

Lemma 4.8 Assume that $(\leftarrow a, \Gamma_0, \Delta_0) \rightsquigarrow_{suc}^e (\Gamma, \Delta)$ for $a \in B_P$. Then

$$[(a^* \notin \Gamma_0 \Rightarrow a^* \notin \Gamma) \wedge (a^* \notin \Delta_0 \Rightarrow a^* \notin \Delta)].$$

Proof See the appendix.

3. Adjustments and abducibles obtained by finitely failing derivations

In case that there is an abductive finitely failing derivation in some derivation, and in relation with the abductive goal, we have:

Lemma 4.9 Assume that $\gamma : (\Gamma_0, \Delta_0) \rightsquigarrow_{any} (\Gamma, \Delta)$ and $\mu : (\leftarrow B, \Gamma_a, \Delta_a) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$ such that $\gamma \gg \mu$. Also assume that $(F, \Gamma_b, \Delta_b) \in \mu$, where $F = \{G_1^*, \dots, G_l^*\}$ for $G_i^* = \leftarrow L_1^i \dots L_{n_i}^i$ ($1 \leq i \leq l$). Then

$$\forall i, \varphi(\text{ground substitution}), \exists j : [(1 \leq i \leq l) \Rightarrow [(1 \leq j \leq n_i) \wedge (L_j^i \varphi = a \in B_P \Rightarrow a \notin S_P(\Gamma^+ \cup \Delta^+)) \wedge (L_j^i \varphi = a^* \in (B_P)^* \Rightarrow a^* \notin \Gamma \cup \Delta)]]$$

Proof See the appendix.

Lemma 4.10 $(F, \Gamma_0, \Delta_0) \rightsquigarrow_{ff} (\Gamma, \Delta)$, where $F = \{G_1^*, \dots, G_l^*\}$ for $G_i^* = \leftarrow L_1^i \dots L_{n_i}^i$ ($1 \leq i \leq l$). Then

$$\forall i, \varphi(\text{ground substitution}), \exists j : [(1 \leq i \leq l) \Rightarrow [(1 \leq j \leq n_i) \wedge (L_j^i \varphi = a \in B_P \Rightarrow a \notin S_P(\overline{S_P(\Delta^+)})) \wedge (L_j^i \varphi = a^* \in (B_P)^* \Rightarrow a \in S_P(\Delta^+))]].$$

4. Adjustments and abducibles obtained by adjusting derivations

In case that there is an adjusting derivation, and in relation with the abductive goal, we have:

Lemma 4.11 Assume that $(C, \Gamma_0, \Delta_0) \rightsquigarrow_{adj} (\Gamma, \Delta)$, where $C = \{(V_1, G_1^*), \dots, (V_l, G_l^*)\}$ for $G_i^* = \leftarrow L_1^i \dots L_{n_i}^i$ ($1 \leq i \leq l$). Then

$$\exists i, \varphi(\text{ground substitution}), \forall j : [(1 \leq i \leq l) \wedge [(1 \leq j \leq n_i) \Rightarrow [(L_j^i \varphi = a \in B_P \Rightarrow a \in S_P(\Gamma^+ \cup \Delta^+)) \wedge (L_j^i \varphi = a^* \in (B_P)^* \Rightarrow a^* \in \Gamma \cup \Delta)]]]]$$

Proof See the appendix.

Lemma 4.12 Assume that $\gamma : (\emptyset, \emptyset) \rightsquigarrow_{any} (\Gamma_0, \Delta_0)$ and $\nu : (\{(t, \leftarrow B)\}, \Gamma_a, \Delta_a) \rightsquigarrow_{adj} (\Gamma'_a, \Delta'_a)$ such that $\gamma \gg \nu$. Let $(C, \Gamma_b, \Delta_b) \in \nu$, where $C = \{(V_1, G_1^*), \dots, (V_l, G_l^*)\}$ for $G_i^* = \leftarrow L_1^i \dots L_{n_i}^i$ ($1 \leq i \leq l$). Then

$$\forall i, \varphi(\text{ground substitution}), \exists j : [(1 \leq i \leq l) \wedge (V_i = t) \Rightarrow [(1 \leq j \leq n_i) \wedge [(L_j^i \varphi = a \in B_P \Rightarrow a \notin S_P(\overline{S_P(\Gamma_0^+ \cup \Delta_0^+)}) \wedge (L_j^i \varphi = a^* \in (B_P)^* \Rightarrow a \in S_P(\Gamma_0^+ \cup \Delta_0^+))]]]]]$$

Proof See the appendix.

For the similar reason in Lemma 4.12, we have:

Lemma 4.13 Assume that $\nu : (\{(t, G^*)\}, \emptyset, \emptyset) \rightsquigarrow_{adj} (\Gamma_0, \Delta_0)$. Let $(C, \Gamma_b, \Delta_b) \in \nu$, where $C = \{(V_1, G_1^*), \dots, (V_l, G_l^*)\}$ for $G_i^* = \leftarrow L_1^i \dots L_{n_i}^i$ ($1 \leq i \leq l$). Then

$$\forall i, \varphi(\text{ground substitution}), \exists j : [(1 \leq i \leq l) \wedge (V_i = t) \Rightarrow [(1 \leq j \leq n_i) \wedge [(L_j^i \varphi = a \in B_P \Rightarrow a \notin S_P(\overline{S_P(\Gamma_0^+ \cup \Delta_0^+)}) \wedge (L_j^i \varphi = a^* \in (B_P)^* \Rightarrow a \in S_P(\Gamma_0^+ \cup \Delta_0^+))]]]]]$$

5. Lemmas for the soundness theorem

Now we have several lemmas, to be made use of, directly for the soundness proof, which are established by means of preceding lemmas.

Lemma 4.14 If there is $\gamma : (\emptyset, \emptyset) \rightsquigarrow_{any} (\Gamma_0, \Delta_0)$, then $\Delta_0^+ \subseteq A_P(\Delta_0^+)$.

Proof

$$\begin{aligned} a^* \in \Delta_0 &\Rightarrow \mu : (\{\leftarrow a\}, \Gamma_a, \Delta_a) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a) \text{ such that } \gamma \gg \mu \\ &\Rightarrow a \notin S_P(\overline{S_P(\Delta_a'^+)}) \text{ (by Lemma 4.10)} \\ &\Rightarrow a \in S_P(\overline{S_P(\Delta_a'^+)}) \\ &\Rightarrow a \in A_P(\Delta_a'^+) \\ &\Rightarrow a \in A_P(\Delta_0^+) \text{ (by } \Delta_a' \subseteq \Delta_0 \text{ and monotonicity of } A_P) \end{aligned}$$

Note that $a^* \in \Delta_0$ iff $a \in \Delta_0^+$. Hence $\Delta_0^+ \subseteq A_P(\Delta_0^+)$.

q.e.d.

Now consider the family of sets obtained by applying A_P inductively.

Definition 4.15 We define Δ_α inductively as follows.

$$\Delta_\alpha^+ = \begin{cases} A_P(\Delta_{\alpha-1}^+) & \text{if } \alpha \text{ is a successor ordinal,} \\ \cup_{\beta < \alpha} A_P(\Delta_\beta^+) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Lemma 4.16 Assume that $\gamma : (\emptyset, \emptyset) \rightsquigarrow_{any} (\Gamma_0, \Delta_0)$. For $X \subseteq B_P$, let

- (1) $Q1(X) \equiv \Gamma_0^+ \subseteq S_P(\Gamma_0^+ \cup X)$,
- (2) $Q2(X) \equiv \Gamma_0^+ \subseteq A_P(\Gamma_0^+ \cup X)$, and
- (3) $Q3(X) \equiv X \cap S_P(\Gamma_0^+ \cup X) = \emptyset$.

Whether $Q = Q1, Q2$, or $Q3$, the following three conditions are satisfied.

- (a) $Q(\Delta_0^+)$.
- (b) $Q(\Delta_\alpha^+)$ implies $Q(A_P(\Delta_\alpha^+))$.
- (c) Q is inclusive, that is,

$$\forall L : [L \subseteq 2^{B_P} \text{ is a chain} \Rightarrow [[\forall Y_\alpha \in L : Q(Y_\alpha)] \Rightarrow Q(\cup L)]].$$

Proof (1) As regards $Q1$:

(a)

$$\begin{aligned} a^* \in \Gamma_0 &\Rightarrow \nu : (\{(t, \leftarrow a)\}, \Gamma_a, \Delta_a) \rightsquigarrow_{adj} (\Gamma'_a, \Delta'_a) \text{ and } \gamma \gg \nu \\ &\Rightarrow a \in S_P(\Gamma_a^+ \cup \Delta_a^+) \text{ (by Lemma 4.11)} \\ &\Rightarrow a \in S_P(\Gamma_0^+ \cup \Delta_0^+) \text{ (by } \Gamma'_a \subseteq \Gamma_0, \Delta'_a \subseteq \Delta_0 \text{ and} \\ &\quad \text{monotonicity of } S_P). \end{aligned}$$

Note that $a^* \in \Gamma_0$ iff $a \in \Gamma_0^+$. Hence $\Gamma_0^+ \subseteq S_P(\Gamma_0^+ \cup \Delta_0^+)$.

(b) Assume $Q(\Delta_\alpha^+)$. By Lemma 4.14, $\Delta_\alpha^+ \subseteq A_P(\Delta_\alpha^+)$. By monotonicity of A_P and by induction, $\Delta_\alpha^+ \subseteq A_P(\Delta_\alpha^+)$. Combining it with monotonicity of S_P ,

$$S_P(\Gamma_0^+ \cup \Delta_\alpha^+) \subseteq S_P(\Gamma_0^+ \cup A_P(\Delta_\alpha^+)).$$

It follows that $\Gamma_0^+ \subseteq S_P(\Gamma_0^+ \cup A_P(\Delta_\alpha^+))$. Hence $Q1$ is preserved under A_P .

(c) For any $Y_\alpha \in L$, $S_P(\Gamma_0^+ \cup Y_\alpha) \subseteq S_P(\Gamma_0^+ \cup (\cup L))$. Clearly $Q1$ is inclusive.

(2) As regards $Q2$:

(a)

$$\begin{aligned} a^* \in \Gamma_0 &\Rightarrow \nu : (\{(t, \leftarrow a)\}, \Gamma_a, \Delta_a) \rightsquigarrow_{adj} (\Gamma'_a, \Delta'_a) \text{ and } \gamma \gg \nu \\ &\Rightarrow a \notin S_P(S_P(\Gamma_0^+ \cup \Delta_0^+)) \text{ (by Lemma 4.12)} \\ &\Rightarrow a \in \overline{S_P(\Gamma_0^+ \cup \Delta_0^+)} \\ &\Rightarrow a \in A_P(\Gamma_0^+ \cup \Delta_0^+) \end{aligned}$$

Note that $a^* \in \Gamma_0$ iff $a \in \Gamma_0^+$. Hence $\Gamma_0^+ \subseteq A_P(\Gamma_0^+ \cup \Delta_0^+)$.

(b) As we see, $\Delta_\alpha^+ \subseteq A_P(\Delta_\alpha^+)$. By monotonicity of A_P and by that $\Delta_\alpha^+ \subseteq A_P(\Delta_\alpha^+)$, $A_P(\Gamma_0^+ \cup \Delta_\alpha^+) \subseteq A_P(\Gamma_0^+ \cup A_P(\Delta_\alpha^+))$. Assume that $Q2(\Delta_\alpha^+)$, that is, $\Gamma_0^+ \subseteq A_P(\Gamma_0^+ \cup \Delta_\alpha^+)$. Then $\Gamma_0^+ \subseteq A_P(\Gamma_0^+ \cup A_P(\Delta_\alpha^+))$.

(c) For any $Y_\alpha \in L$, $A_P(\Gamma_0^+ \cup Y_\alpha) \subseteq A_P(\Gamma_0^+ \cup (\cup L))$. Clearly $Q2$ is inclusive.

(3) As regards $Q3$:

(a)

$$\begin{aligned} a^* \in \Delta_0 &\Rightarrow \mu : (\{\leftarrow a\}, \Gamma_a, \Delta_a) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a) \text{ and } \gamma \gg \mu \\ &\Rightarrow a \notin S_P(\Gamma_0^+ \cup \Delta_0^+). \text{ (by Lemma 4.9)} \end{aligned}$$

Note that $a^* \in \Delta_0$ iff $a \in \Delta_0^+$. Hence $\Delta_0^+ \cap S_P(\Gamma_0^+ \cup \Delta_0^+) = \emptyset$.

(b) Since $\Delta_\alpha^+ \subseteq A_P(\Delta_\alpha^+)$, it is easy to see that $\Delta_0 \subseteq \Delta_\alpha$. Because S_P is monotonic,

$$\begin{aligned} Q3(\Delta_\alpha^+) &\Rightarrow \Delta_\alpha^+ \cap S_P(\Gamma_0^+ \cup \Delta_\alpha^+) = \emptyset \\ &\Rightarrow \Delta_\alpha^+ \subseteq \overline{S_P(\Gamma_0^+ \cup \Delta_\alpha^+)} \\ &\Rightarrow S_P(\Delta_\alpha^+) \subseteq \overline{S_P(S_P(\Gamma_0^+ \cup \Delta_\alpha^+))} \\ &\Rightarrow \overline{S_P(S_P(\Gamma_0^+ \cup \Delta_\alpha^+))} \subseteq \overline{S_P(\Delta_\alpha^+)} \\ &\Rightarrow A_P(\Gamma_0^+ \cup \Delta_\alpha^+) \subseteq S_P(\Delta_\alpha^+). \end{aligned}$$

By (2) and monotonicity of A_P ,

$$\Gamma_0^+ \subseteq A_P(\Gamma_0^+ \cup \Delta_\alpha^+) \subseteq A_P(\Gamma_0^+ \cup \Delta_\alpha^+) \text{ and } A_P(\Delta_\alpha^+) \subseteq A_P(\Gamma_0^+ \cup \Delta_\alpha^+).$$

It follows that $\Gamma_0^+ \cup A_P(\Delta_\alpha^+) \subseteq A_P(\Gamma_0^+ \cup \Delta_\alpha^+)$. By using monotonicity of S_P ,

$$\begin{aligned} Q3(\Delta_\alpha^+) &\Rightarrow A_P(\Gamma_0^+ \cup \Delta_\alpha^+) \subseteq \overline{S_P(\Delta_\alpha^+)} \\ &\Rightarrow \Gamma_0^+ \cup A_P(\Delta_\alpha^+) \subseteq \overline{S_P(\Delta_\alpha^+)} \\ &\Rightarrow \overline{S_P(\Gamma_0^+ \cup A_P(\Delta_\alpha^+))} \subseteq \overline{S_P(\overline{S_P(\Delta_\alpha^+)})} \\ &\Rightarrow \overline{S_P(\overline{S_P(\Delta_\alpha^+)})} \subseteq \overline{S_P(\Gamma_0^+ \cup A_P(\Delta_\alpha^+))} \\ &\Rightarrow A_P(\Delta_\alpha^+) \subseteq \overline{S_P(\Gamma_0^+ \cup A_P(\Delta_\alpha^+))} \\ &\Rightarrow Q3(A_P(\Delta_\alpha^+)). \end{aligned}$$

Hence $Q3$ is preserved under A_P .

(c) Assume that $Q3$ is not inclusive, and that $Q3(\cup L)$ does not hold for some L even if $Q3(Y_\alpha)$ for any $Y_\alpha \in L$. Then

$$\begin{aligned} &\exists a : [a \in \cup L \wedge a \in S_P(\Gamma_0^+ \cup (\cup L))] \\ \Rightarrow &\exists Y_\alpha \in L : [a \in Y_\alpha \wedge a \in S_P(\Gamma_0^+ \cup Y_\alpha)], \end{aligned}$$

which contradicts $Q3(Y_\alpha)$. Thus $Q3$ is inclusive. q.e.d.

Lemma 4.17 Assume $\gamma : (\emptyset, \emptyset) \rightsquigarrow_{any} (\Gamma_0, \Delta_0)$. Then there exists Δ satisfying the following conditions:

- (1) $\Delta_0 \subseteq \Delta$.
- (2) $\Delta^+ = A_P(\Delta^+)$.
- (3) $\Gamma_0^+ \subseteq S_P(\Gamma_0^+ \cup \Delta^+)$.
- (4) $\Gamma_0^+ \subseteq A_P(\Gamma_0^+ \cup \Delta^+)$.
- (5) $\Delta^+ \cap S_P(\Gamma_0^+ \cup \Delta^+) = \emptyset$.

Proof (1) and (2): Let Δ_α^+ be defined as in Definition 4.15. By monotonicity of A_P and Lemma 4.14, there exists γ such that $\Delta_0^+ \subseteq \Delta_\gamma^+$ and $\Delta_\gamma^+ = A_P(\Delta_\gamma^+)$.

(3), (4) and (5): These follow from Lemma 4.16 and fixpoint induction. q.e.d.

Lemma 4.18 Assume $\gamma : (\emptyset, \emptyset) \rightsquigarrow_{any} (\Gamma_0, \Delta_0)$. Then there exists Δ satisfying the following conditions:

- (1) $\Delta_0 \subseteq \Delta$ and $\Delta^+ = A_P(\Delta^+)$.
- (2) $\Delta^+ \cap S_P(\Delta^+) = \emptyset$.
- (3) $\Gamma_0 \cap \Delta = \emptyset$.
- (4) $\Gamma_0^+ \cap S_P(\Delta^+) = \emptyset$.

Proof By Lemma 4.17, there exists Δ satisfying:

- (a) $\Delta_0 \subseteq \Delta$ and $\Delta^+ = A_P(\Delta^+)$.
- (b) $\Gamma_0^+ \subseteq S_P(\Gamma_0^+ \cup \Delta^+)$.
- (c) $\Gamma_0^+ \subseteq A_P(\Gamma_0^+ \cup \Delta^+)$.
- (d) $\Delta^+ \cap S_P(\Gamma_0^+ \cup \Delta^+) = \emptyset$.

Hence (1) is satisfied. From (d) and from that $S_P(\Delta^+) \subseteq S_P(\Gamma_0^+ \cup \Delta^+)$, (2) follows. By (b)

and (d), (3) holds. It follows from (c) and (d) that

$$\begin{aligned} \Gamma_0^+ \subseteq A_P(\Gamma_0^+ \cup \Delta^+) &\Rightarrow \Gamma_0^+ \subseteq \overline{S_P(S_P(\Gamma_0^+ \cup \Delta^+))} \\ &\Rightarrow \Gamma_0^+ \cap S_P(\overline{S_P(\Gamma_0^+ \cup \Delta^+)}) = \emptyset, \text{ and} \\ \Delta^+ \cap S_P(\Gamma_0^+ \cup \Delta^+) = \emptyset &\Rightarrow \Delta^+ \subseteq \overline{S_P(\Gamma_0^+ \cup \Delta^+)} \\ &\Rightarrow S_P(\Delta^+) \subseteq S_P(\overline{S_P(\Gamma_0^+ \cup \Delta^+)}) \end{aligned}$$

Hence (4) is satisfied.

q.e.d.

6. Soundness theorems

Finally we have the soundness theorems of the newly presented abductive proof procedure.

Theorem 4.19 *Let $\langle P^*, Ab^*, I_t^* \rangle$ be an abduction framework. Assume $\gamma : (\emptyset, \emptyset) \rightsquigarrow_{any} (\Gamma_0, \Delta_0)$. Then there exist Γ and Δ such that $\Gamma_0 \subseteq \Gamma$, $\Delta_0 \subseteq \Delta$ and $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* .*

Proof It follows from Lemma 4.18 that there exists Δ such that (1) $\Delta_0 \subseteq \Delta$, $\Delta^+ = A_P(\Delta^+)$, (2) $\Delta^+ \cap S_P(\Delta^+) = \emptyset$, (3) $\Gamma_0 \cap \Delta = \emptyset$, and (4) $\Gamma_0^+ \cap S_P(\Delta^+) = \emptyset$. By (3) and (4), $\Gamma_0^+ \cap (\Delta^+ \cup S_P(\Delta^+)) = \emptyset$. By setting $\Gamma^+ = \Delta^+ \cup S_P(\Delta^+)$, we have $\Gamma_0 \subseteq \Gamma$. By using Theorem 3.16, $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* . q.e.d.

Theorem 4.20 *Let $\langle P^*, Ab^*, I_t^* \rangle$ be an abduction framework. If $((\leftarrow L_1 \dots L_n), \emptyset, \emptyset) \rightsquigarrow_{suc}^{\theta} (\Gamma_0, \Delta_0)$, then there exist Γ and Δ such that:*

- (1) $\Gamma_0 \subseteq \Gamma$, $\Delta_0 \subseteq \Delta$ and $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* .
- (2) $P^* \cup \Gamma^u \cup \Delta \models_3 \forall((L_1 \wedge \dots \wedge L_n)\theta)$.

Proof By Lemma 4.4, it is obtained that for any i ($1 \leq i \leq n$)

$L_i\theta\varphi = a \in B_P$ implies $a \in S_P(\Delta_0^+)$, and

$L_i\theta\varphi = a^* \in (B_P)^*$ implies $a^* \in \Delta_0$.

By Lemma 4.18, $\Gamma_0 \cap \Delta_0 = \emptyset$. It follows from Corollary 3.9 that for any i ($1 \leq i \leq n$), $L_i\theta\varphi \in B_{P^*}$ implies $P^* \cup \Gamma_0^u \cup \Delta_0 \models_3 L_i\theta\varphi$. Hence for any ground substitution φ , $P^* \cup \Gamma_0^u \cup \Delta_0 \models_3 (L_1 \wedge \dots \wedge L_n)\theta\varphi$. Finally

$$P^* \cup \Gamma_0^u \cup \Delta_0 \models_3 \forall((L_1 \wedge \dots \wedge L_n)\theta).$$

By Theorem 4.19, there exist Γ and Δ such that $\Gamma_0 \subseteq \Gamma$, $\Delta_0 \subseteq \Delta$ and $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* , where $\Gamma^+ = \Delta^+ \cup \overline{S_P(\Delta^+)}$. By monotonicity of S_P , $S_P(\Delta_0^+) \subseteq S_P(\Delta^+)$. Owing to Corollary 3.9 and similar reasons as above, we have:

$$P^* \cup \Gamma^u \cup \Delta \models_3 \forall((L_1 \wedge \dots \wedge L_n)\theta).$$

q.e.d.

Theorem 4.21 *Assume an abduction framework $\langle P^*, Ab^*, I_t^* \rangle$. If $(\{\leftarrow L_1 \dots L_n\}, \emptyset, \emptyset) \rightsquigarrow_{ff} (\Gamma_0, \Delta_0)$, then there exist Γ and Δ such that:*

- (1) $\Gamma_0 \subseteq \Gamma$, $\Delta_0 \subseteq \Delta$ and $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* .
- (2) $P^* \cup \Gamma^u \cup \Delta \not\models_3 \forall((L_1 \wedge \dots \wedge L_n) \leftarrow u)$.

Proof By the proof of Theorem 4.19, there exist Γ and Δ such that $\Gamma_0 \subseteq \Gamma$, $\Delta_0 \subseteq \Delta$, $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* , and $\Gamma^+ = \Delta^+ \cup \overline{S_P(\Delta^+)}$. Now assume that there exists a ground substitution φ such that

$$P^* \cup \Gamma^u \cup \Delta \models_3 (L_1 \wedge \dots \wedge L_n) \varphi \leftarrow \mathbf{u}.$$

Then $\forall i : [(1 \leq i \leq n) \Rightarrow [P^* \cup \Gamma^u \cup \Delta \models_3 L_i \varphi \leftarrow \mathbf{u}]]$. By Lemma 3.14,

$$\forall i : [(1 \leq i \leq n) \Rightarrow [(L_i \varphi = a \in B_P \Rightarrow a \in S_P(\Gamma^+ \cup \Delta^+)) \wedge (L_i \varphi = a^* \in (B_P)^* \Rightarrow a^* \in \Gamma \cup \Delta)]]]. \quad (\#)$$

By Lemma 4.10,

$$\exists j : [(1 \leq j \leq n) \wedge [(L_j \varphi = a \in B_P \Rightarrow a \notin S_P(\overline{S_P(\Delta_0^+)}) \wedge (L_j \varphi = a^* \in (B_P)^* \Rightarrow a \in S_P(\Delta_0^+))]]].$$

In case that $L_j \varphi = a \in B_P$, $a \notin S_P(\overline{S_P(\Delta_0^+)})$. By Lemma 3.15 and $(\#)$,

$$a \in S_P(\Gamma^+ \cup \Delta^+) = S_P(\overline{S_P(\Delta^+)}).$$

Also

$$\begin{aligned} \Delta_0^+ \subseteq \Delta^+ &\Rightarrow S_P(\Delta_0^+) \subseteq S_P(\Delta^+) \\ &\Rightarrow \overline{S_P(\Delta^+)} \subseteq \overline{S_P(\Delta_0^+)} \\ &\Rightarrow S_P(\overline{S_P(\Delta^+)}) \subseteq S_P(\overline{S_P(\Delta_0^+)}). \end{aligned}$$

It follows that $a \notin S_P(\overline{S_P(\Delta_0^+)}) \Rightarrow a \notin S_P(\overline{S_P(\Delta^+)})$, which is a contradiction.

In case that $L_j \varphi = a^* \in (B_P)^*$, $a \in S_P(\Delta_0^+)$. By Lemma 3.15, $a \in \Gamma^+ \cup \Delta^+ = \overline{S_P(\Delta^+)}$. Also $a \in S_P(\Delta_0^+) \subseteq S_P(\Delta^+)$. This is a contradiction. q.e.d.

Theorem 4.22 Assume an abduction framework $\langle P^*, Ab^*, I_t^* \rangle$. If $(\{(t, \leftarrow L_1 \dots L_n)\}, \emptyset, \emptyset) \rightsquigarrow_{adj} (\Gamma_0, \Delta_0)$, then there exist Γ and Δ such that:

- (1) $\Gamma_0 \subseteq \Gamma$, $\Delta_0 \subseteq \Delta$ and $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* .
- (2) $P^* \cup \Gamma^u \cup \Delta \models_3 \exists((L_1 \wedge \dots \wedge L_n) \leftarrow \mathbf{u})$ and $P^* \cup \Gamma^u \cup \Delta \not\models_3 \forall(L_1 \wedge \dots \wedge L_n)$.

Proof By the proof of Theorem 4.19, there exist Γ and Δ such that $\Gamma_0 \subseteq \Gamma$, $\Delta_0 \subseteq \Delta$, $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* , and $\Gamma^+ = \overline{\Delta^+ \cup S_P(\Delta^+)}$.

By Lemma 4.11,

$$\exists \varphi(\text{ground substitution}), \forall i : [(1 \leq i \leq n) \Rightarrow [(L_i \varphi = a \in B_P \Rightarrow a \in S_P(\Gamma_0^+ \cup \Delta_0^+)) \wedge (L_i \varphi = a^* \in (B_P)^* \Rightarrow a^* \in \Gamma_0 \cup \Delta_0)]]].$$

By Lemma 4.18, $\Gamma_0 \cap \Delta_0 = \emptyset$. By Corollary 3.11,

$$\exists \varphi(\text{groundsubstitution}), \forall i : [(1 \leq i \leq n) \Rightarrow [P^* \cup \Gamma_0^u \cup \Delta_0 \models_3 L_i \varphi \leftarrow \mathbf{u}]].$$

It follows that $\exists \varphi(\text{groundsubstitution}) : [P^* \cup \Gamma_0^u \cup \Delta_0 \models_3 (L_1 \wedge \dots \wedge L_n) \varphi \leftarrow \mathbf{u}]$. Hence

$$P^* \cup \Gamma_0^u \cup \Delta_0 \models_3 \exists((L_1 \wedge \dots \wedge L_n) \leftarrow \mathbf{u}).$$

Since S_P is monotonic, $S_P(\Gamma_0^+ \cup \Delta_0^+) \subseteq S_P(\Gamma^+ \cup \Delta^+)$. By the similar discussion as above,

$$P^* \cup \Gamma^u \cup \Delta \models_3 \exists((L_1 \wedge \dots \wedge L_n) \leftarrow \mathbf{u}).$$

Now assume that there exists a ground substitution φ such that $P^* \cup \Gamma^u \cup \Delta \models_3 (L_1 \wedge \dots \wedge L_n) \varphi$. Then $\forall i : [(1 \leq i \leq n) \Rightarrow [P^* \cup \Gamma^u \cup \Delta \models_3 L_i \varphi]]$. By Corollary 3.9,

$$\forall i : [(1 \leq i \leq n) \Rightarrow [(L_i \varphi = a \in B_P \Rightarrow a \in S_P(\Delta^+)) \wedge (L_i \varphi = a^* \in (B_P)^* \Rightarrow a^* \in \Delta)]]].$$

By Lemma 4.13,

$$\begin{aligned} \exists j : [(1 \leq j \leq n) \wedge \\ [(L_j\varphi = a \in B_P \Rightarrow a \notin S_P(\overline{S_P(\Gamma_0^+ \cup \Delta_0^+)}) \\ \wedge (L_j\varphi = a^* \in (B_P)^* \Rightarrow a \in S_P(\Gamma_0^+ \cup \Delta_0^+))]]]. \end{aligned}$$

In case that $L_j\varphi = a \in B_P$,

$$a \notin S_P(\overline{S_P(\Gamma_0^+ \cup \Delta_0^+)}) \Leftrightarrow a \in A_P(\Gamma_0^+ \cup \Delta_0^+) \subseteq A_P(\Gamma_0^+ \cup \Delta^+).$$

By Lemma 4.17, $\Delta^+ \cap S_P(\Gamma_0^+ \cup \Delta^+) = \emptyset$. It follows that

$$\begin{aligned} \Delta^+ \cap S_P(\Gamma_0^+ \cup \Delta^+) = \emptyset &\Rightarrow \Delta^+ \subseteq \overline{S_P(\Gamma_0^+ \cup \Delta^+)} \\ &\Rightarrow S_P(\Delta^+) \subseteq \overline{S_P(\Gamma_0^+ \cup \Delta^+)} \\ &\Rightarrow S_P(\Delta^+) \subseteq \overline{A_P(\Gamma_0^+ \cup \Delta^+)} \\ &\Rightarrow S_P(\Delta^+) \cap A_P(\Gamma_0^+ \cup \Delta^+) = \emptyset. \end{aligned}$$

But we see that $a \in S_P(\Delta^+) \cap A_P(\Gamma_0^+ \cup \Delta^+)$, which is a contradiction.

In case that $L_j\varphi = a^* \in (B_P)^*$, by Lemma 4.17, $\Delta^+ \cap S_P(\Gamma_0^+ \cup \Delta^+) = \emptyset$. But we see that $a \in \Delta^+$ and $a \in S_P(\Gamma_0^+ \cup \Delta_0^+) \subseteq S_P(\Gamma_0^+ \cup \Delta^+)$. It is a contradiction. q.e.d.

Example 4.23 Assume a general logic program P as in Example 3.2. As we have seen in Example 4.1, we see that $(\leftarrow t, \emptyset, \emptyset) \rightsquigarrow_{suc}^{\epsilon} (\{q^*\}, \{r^*\})$. For $\Gamma = \{p^*, q^*\}$ and $\Delta = \{s^*, r^*\}$, $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* , and $P^* \cup \Gamma^u \cup \Delta \models_3 t$.

5 Concluding Remarks

The primary contributions of the present paper are to have a new integrity constraint in an abduction framework based on general logic programs, and to augment an adjusting derivation to Eshghi and Kowalski procedure in correlation with the presented constraint. As an advantage of the augmented abductive proof procedure, each derivation can be regarded as a start, whether it is an abductive succeeding, finitely failing or adjusting derivation. Hence, as a prerequisite, an adjusting derivation can be implemented before starting a succeeding derivation, to avoid an unsuccessful derivation. The possibility of adopting a general negation as failure, discussed as in [19, 20, 21], for the abduction framework with the presented integrity constraint, and for the augmented abductive proof procedure, may be studied.

In [2, 3], positive and negative loops are examined in computations with respect to the well-founded semantics. In [18], a loop is detected by an oracle in computations with respect to the well-founded semantics. In more general, a sound and complete derivation is studied in [15]. On the other hand, we present a derivation to detect the undefined as a logical consequence. As regards the completeness of the presented proof procedure, we have a question of which class of programs as theories can be a subject.

Acknowledgement

The work was supported by the Telecommunications Advancement Foundation.

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Appendix

A.1 Proofs in Section 3

Proof of Lemma 3.5

$$\begin{aligned}
 a \in S_P(\Delta^+) &\Leftrightarrow a \in B_P \cap T_{P\Delta^+} \uparrow \omega && \text{(by the definition of } S_P) \\
 &\Leftrightarrow a \in B_P \cap T_{P^* \cup \Delta} \uparrow \omega && \text{(by the definition of } P^{\Delta^+}) \\
 &\Leftrightarrow a \in B_P \cap \{b \in B_{P^* \cup \Delta} \mid P^* \cup \Delta \models b\} && \text{(as in [14])} \\
 &\Leftrightarrow a \in \{b \in B_P \mid P^* \cup \Delta \models b\}
 \end{aligned}$$

Hence we have the lemma.

q.e.d.

Proof of Lemma 3.6 Assume that $a^* \in \Delta$. It is evident that a^* is true in any 2-valued Herbrand model of $P^* \cup \Delta$. On the other hand, assume that $P^* \cup \Delta \models a^*$. Because a^* does not appear in any head of the clause of $ground(P^*)$, $a^* \in \Delta$ as long as $P^* \cup \Delta \models a^*$. q.e.d.

Proof of Lemma 3.7 (1) It is shown in [19] that $P \models a$ iff $P \models_3 a$ if P is a set of definite clauses. The lemma follows it. q.e.d.

Proof of Lemma 3.8 (1)(i) Because $\Delta \subseteq S_P(\Delta^+) \cup \Delta$, M is a model of Δ .

(ii) For each clause $a \leftarrow L_1 \dots L_n \in ground(P^*)$:

(a) In case that $a \in S_P(\Delta^+)$, a is true in M , and thus $a \leftarrow L_1 \dots L_n$ is true in M .

(b) In case that $a \notin S_P(\Delta^+)$, a is regarded as **u** (the undefined). It follows from the definition of S_P that $L_i = b \in B_P$ but $b \notin S_P(\Delta^+)$, or $L_i = b^* \in (B_P)^*$ but $b^* \notin \Delta$ for some L_i ($1 \leq i \leq n$). Hence $L_i \notin S_P(\Delta^+) \cup \Delta$, and L_i is evaluated as **u** by M . Finally $a \leftarrow L_1 \dots L_n$ is true in M .

(iii) $\Gamma \cap \Delta = \emptyset$ implies $\Gamma \cap (S_P(\Delta^+) \cup \Delta) = \emptyset$. It follows that any member in Γ is interpreted as **u** in M . That is, M is a model of Γ^u .

By (i), (ii) and (iii), M is a model of $P^* \cup \Gamma^u \cup \Delta$.

(2) (i) Assume that $P^* \cup \Delta \models_3 l$. Since any 3-valued Herbrand model of $P^* \cup \Gamma^u \cup \Delta$ is also a 3-valued Herbrand model of $P^* \cup \Delta$, l is true in any 3-valued Herbrand model of $P^* \cup \Gamma^u \cup \Delta$. It follows that $P^* \cup \Gamma^u \cup \Delta \models_3 l$.

(ii) Assume that $P^* \cup \Gamma^u \cup \Delta \models_3 l$. Assume that $P^* \cup \Delta \not\models_3 l$. By Lemma 3.7, $P^* \cup \Delta \not\models l$. If $l \in B_P$ then $l \notin S_P(\Delta^+)$ by Lemma 3.5. If $l \in (B_P)^*$ then $l \notin \Delta$ by Lemma 3.6. Hence

$l \notin S_P(\Delta^+) \cup \Delta$. l is evaluated as $\mathbf{u}$ by $M = \langle S_P(\Delta^+) \cup \Delta, \emptyset \rangle$, which is a 3-valued Herbrand model of $P^* \cup \Gamma^u \cup \Delta$. This contradicts that $P^* \cup \Gamma^u \cup \Delta \models_3 l$. q.e.d.

Proof of Lemma 3.10 (1) Assume that $\langle IT, IF \rangle$ is a 3-valued Herbrand model of $P^* \cup \Gamma^u \cup \Delta$. Let $I = B_{P^*} - IF$.

- (i) Because $\Delta \subseteq IT$, I is a 2-valued Herbrand model of Δ .
- (ii) Since $\langle IT, IF \rangle$ is a 3-valued Herbrand model of Γ^u , $a^* \in B_{P^*} - (IT \cup IF)$ if $a^* \leftrightarrow \mathbf{u} \in \Gamma^u$. Hence $a^* \in I$. That is, $\Gamma \subseteq I$, and I is a 2-valued Herbrand model of Γ .
- (iii) Since $\langle IT, IF \rangle$ is a 3-valued Herbrand model of P^* , any ground clause obtained from a clause of P^* is interpreted by $\langle IT, IF \rangle$ as follows:

$$t \leftarrow t, t \leftarrow \mathbf{u}, \mathbf{u} \leftarrow \mathbf{u}, t \leftarrow f, \mathbf{u} \leftarrow f, \text{ or } f \leftarrow f.$$

Hence, by I , any ground clause obtained from a clause of P^* is interpreted as

$$t \leftarrow t, t \leftarrow t, t \leftarrow t, t \leftarrow f, t \leftarrow f, \text{ or } f \leftarrow f,$$

respectively. It follows that I is a 2-valued Herbrand model of P^* .

(2) Assume that I is a 2-valued Herbrand model of $P^* \cup \Gamma \cup \Delta$. Since I is a 2-valued Herbrand model of $P^* \cup \Gamma \cup \Delta$, $\Gamma \subseteq I$ and $\Delta \subseteq I$. Let $M = \langle I - \Gamma, B_{P^*} - I \rangle$.

- (i) Because $\Gamma \cap \Delta = \emptyset$, $\Delta \subseteq I - \Gamma$, and M is a 3-valued Herbrand model of Δ .
- (ii) Assume that $a^* \in \Gamma$, that is, $a^* \leftrightarrow \mathbf{u} \in \Gamma^u$. It follows that $a^* \notin I - \Gamma$ and $a^* \notin B_{P^*} - I$. Hence a^* is interpreted as $\mathbf{u}$ by M . That is, M is a 3-valued Herbrand model of Γ^u .
- (iii) Since I is a 2-valued Herbrand model of P^* , any ground clause obtained from a clause of P^* is interpreted by I as:

$$t \leftarrow t, t \leftarrow f, \text{ or } f \leftarrow f.$$

Note that any head of the ground clause is not in Γ . By M , it is interpreted as

$$t \leftarrow t \text{ or } t \leftarrow \mathbf{u}, t \leftarrow f, \text{ or } f \leftarrow f,$$

respectively. Hence any ground clause obtained from a clause of P^* is interpreted as $\mathbf{t}$ by M . That is, M is a 3-valued Herbrand model of P^* .

By (i), (ii) and (iii), M is a 3-valued Herbrand model of $P^* \cup \Gamma^u \cup \Delta$.

(3) (i) Assume that there is $l \in B_{P^*}$ such that $P^* \cup \Gamma^u \cup \Delta \models_3 l \leftarrow \mathbf{u}$ and $P^* \cup \Gamma \cup \Delta \not\models l$. It follows that there is a 2-valued Herbrand model I of $P^* \cup \Gamma \cup \Delta$, which interprets l as false. By (2), $M = \langle I - \Gamma, B_{P^*} - I \rangle$ is a 3-valued Herbrand model of $P^* \cup \Gamma^u \cup \Delta$. Because $l \notin I$, $l \in B_{P^*} - I$. Thus l is false in M , which contradicts that $P^* \cup \Gamma^u \cup \Delta \models_3 l \leftarrow \mathbf{u}$.

(ii) Assume that there is $l \in B_{P^*}$ such that $P^* \cup \Gamma \cup \Delta \models l$ and $P^* \cup \Gamma^u \cup \Delta \not\models_3 l \leftarrow \mathbf{u}$. It follows that there is a 3-valued Herbrand model $\langle IT, IF \rangle$ of $P^* \cup \Gamma^u \cup \Delta$, which interprets l as false. By (1), $I = B_{P^*} - IF$ is a 2-valued Herbrand model of $P^* \cup \Gamma \cup \Delta$. Because $l \in IF$, $l \notin I$. Thus l is false in I , which contradicts that $P^* \cup \Gamma \cup \Delta \models l$.

By (i) and (ii), $P^* \cup \Gamma^u \cup \Delta \models_3 l \leftarrow \mathbf{u}$ iff $P^* \cup \Gamma \cup \Delta \models l$. q.e.d.

Proof Lemma 3.12 (1) It follows from Lemma 3.6 and Corollary 3.9.

(2) (i) Assume that $s^* \in \Gamma$. In any 3-valued Herbrand model of $P^* \cup \Gamma^u \cup \Delta$, which is a model of Γ^u , $s^* \leftrightarrow \mathbf{u}$ is true. That is, $P^* \cup \Gamma^u \cup \Delta \models_3 s^* \leftrightarrow \mathbf{u}$.

(ii) Assume that $P^* \cup \Gamma^u \cup \Delta \models_3 s^* \leftrightarrow \mathbf{u}$. Because $\Gamma \cap \Delta = \emptyset$, and there is no clause of P^* , whose head contains s^* , s^* should be in Γ .

(3) (i) If $s^* \notin \Gamma \cup \Delta$, then there is a 3-valued Herbrand model of $P^* \cup \Gamma^u \cup \Delta$, which interprets s^* as false. It follows that $P^* \cup \Gamma^u \cup \Delta \not\models_3 s^* \leftarrow u$.

(ii) If $P^* \cup \Gamma^u \cup \Delta \not\models_3 s^* \leftarrow u$, then there is a 3-valued Herbrand model of $P^* \cup \Gamma^u \cup \Delta$, which interprets s^* as false. It follows that $P^* \cup \Gamma^u \cup \Delta \not\models_3 s^*$ and $P^* \cup \Gamma^u \cup \Delta \not\models_3 s^* \leftrightarrow u$. By (1) and (2), $s^* \notin \Gamma$ and $s^* \notin \Delta$. q.e.d.

Proof of Lemma 3.13 (1) Assume that $a \in \mathbf{T}$, that is, $P^* \cup \Gamma^u \cup \Delta \models_3 a$. Since $P^* \cup \Gamma^u \cup \Delta$ satisfies I_t^* , $P^* \cup \Gamma^u \cup \Delta \not\models_3 a \wedge (a^* \leftarrow u)$. It follows that a^* is false in some 3-valued Herbrand model of $P^* \cup \Gamma^u \cup \Delta$. Hence $a^* \in \mathbf{F}$. On the other hand, if $a^* \in \mathbf{F}$, that is, $P^* \cup \Gamma^u \cup \Delta \not\models_3 a^* \leftarrow u$, then $P^* \cup \Gamma^u \cup \Delta \models_3 a$, that is, $a \in \mathbf{T}$, because of the constraint. Hence $a^* \in \mathbf{F}$ implies $a \in \mathbf{T}$.

(2) If $a \in \mathbf{F}$, that is, $P^* \cup \Gamma^u \cup \Delta \not\models_3 a \leftarrow u$, then $P^* \cup \Gamma^u \cup \Delta \models a^*$, because of the constraint. Hence $a^* \in \mathbf{T}$. On the other hand, on the assumption that $a^* \in \mathbf{T}$, $P^* \cup \Gamma^u \cup \Delta \not\models_3 a \leftarrow u$, because of the constraint. Hence $a \in \mathbf{F}$.

(3) It follows from (1) and (2) that

$$a \in \mathbf{U} \Leftrightarrow a \notin \mathbf{T} \wedge a \notin \mathbf{F} \Leftrightarrow a^* \notin \mathbf{T} \wedge a^* \notin \mathbf{F} \Leftrightarrow a^* \in \mathbf{U}.$$

q.e.d.

Proof of Lemma 3.14 By Lemma 3.4, $\Gamma \cap \Delta = \emptyset$.

(1) It follows from Lemmas 3.12, 3.13 and Corollary 3.9.

(2) It follows from Lemmas 3.12, 3.13 and Corollary 3.11.

(3) It follows from (1), (2) and $\Gamma \cap \Delta = \emptyset$ that

$$a^* \in \mathbf{U} \Leftrightarrow a^* \notin \mathbf{T} \cup \mathbf{F} \Leftrightarrow (a^* \in \Gamma \cup \Delta) \wedge (a^* \notin \Delta) \Leftrightarrow a^* \in \Gamma.$$

By (1) and (2), $a^* \in \Gamma \Leftrightarrow a \in S_P(\Gamma^+ \cup \Delta^+) \wedge a \notin S_P(\Delta^+)$.

q.e.d.

A.2 Proofs in Section 4

Proof of Lemma 4.4 It is proved by induction on length k of the derivation: $(\leftarrow L_1 \dots L_n, \Gamma_0, \Delta_0) \rightsquigarrow_{suc} (\theta, \Gamma, \Delta)$.

(1) If $k = 0$, then $n = 0$ and it holds.

(2) Assume that it holds if $k = m$. Let $k = m + 1$. Also assume that a literal L_s is selected in the abductive goal $\leftarrow L_1 \dots L_n$.

(i) In case of (suc1): Consider the case that $L_s = A$ (an atom) and there is $A' \leftarrow L'_1 \dots L'_l \in P^*$ such that $\theta_1 = mgu(A, A')$, and

$$(\leftarrow (L_1 \dots L_{s-1} L'_1 \dots L'_l L_{s+1} \dots L_n) \theta_1, \Gamma_0, \Delta_0) \rightsquigarrow_{suc}^{\theta'} (\Gamma, \Delta)$$

of length m , where $\theta_1 \theta' = \theta$. By induction hypothesis,

$$\begin{aligned} \forall i : & [(1 \leq i \leq n) \wedge (i \neq s) \Rightarrow \\ & [(L_i \theta_1 \theta' \varphi = a \in B_P \Rightarrow a \in S_P(\Delta^+)) \\ & \wedge (L_i \theta_1 \theta' \varphi = a^* \in (B_P)^* \Rightarrow a^* \in \Delta)]] \text{ and} \\ \forall j : & [(1 \leq j \leq l) \Rightarrow [(L'_j \theta_1 \theta' \varphi = a \in B_P \Rightarrow a \in S_P(\Delta^+)) \\ & \wedge (L'_j \theta_1 \theta' \varphi = a^* \in (B_P)^* \Rightarrow a^* \in \Delta)]]]. \end{aligned}$$

Hence $L_s \theta \varphi = A' \theta_1 \theta' \varphi \in S_P(\Delta^+)$ if $A' \theta_1 \theta' \varphi \in B_P$. This completes the induction.

(ii) In case of (suc2): Consider the case that $L_s = A^*$ (ground), $A^* \in \Delta_0$, and

$$(\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n, \Gamma_0, \Delta_0) \rightsquigarrow_{suc} (\theta, \Gamma, \Delta)$$

of length m . By induction hypothesis,

$$\forall i : [(1 \leq i \leq n, i \neq s) \Rightarrow [(L_i \theta \varphi = a \in B_P \Rightarrow a \in S_P(\Delta^+)) \wedge (L_i \theta \varphi = a^* \in (B_P)^* \Rightarrow a^* \in \Delta)]]].$$

Because $A^* \in \Delta_0 \subseteq \Delta$, the induction step is completed.

(iii) In case of (suc3): Consider the case that $L_s = A^*$ (ground), $A^* \notin \Gamma_0 \cup \Delta_0$ and $(\{\leftarrow A\}, \Gamma_0, \Delta_0 \cup \{A^*\}) \rightsquigarrow_{ff} (\Gamma'_0, \Delta'_0)$, then

$$(\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n, \Gamma'_0, \Delta'_0) \rightsquigarrow_{suc} (\theta, \Gamma, \Delta)$$

of length m . By induction hypothesis for the derivation of length m and by that $A^* \in \Delta'_0 \subseteq \Delta$, the induction step is completed. q.e.d.

Proof of Lemma 4.5 It is proved by induction on length k of the derivation: $(G^*, \Gamma_0, \Delta_0) \rightsquigarrow_{suc} (\theta, \Gamma, \Delta)$.

(1) If $k = 0$, then $G^* = \square$ and it holds.

(2) Assume that it holds for $k = m$. Let $k = m + 1$. Also assume that L_s is the selected atom in the abductive goal $\leftarrow L_1 \dots L_n$.

(i) In case of (suc1): Consider the case that $L_s = A$ and there is $A' \leftarrow L'_1 \dots L'_l \in P^*$ such that $\theta_1 = mgu(A, A')$, and $(G_1^*, \Gamma_0, \Delta_0) \rightsquigarrow_{suc} (\theta', \Gamma, \Delta)$ of length m , where

$$G_1^* = \leftarrow (L_1 \dots L_{s-1} L'_1 \dots L'_l L_{s+1} \dots L_n) \theta_1$$

and $\theta_1 \theta' = \theta$. By induction hypothesis,

$$\begin{aligned} \forall \Gamma', \Delta', \Gamma'', \Delta'' : [(\Gamma \subseteq \Gamma') \wedge (\Delta \subseteq \Delta') \Rightarrow \\ (G_1^*, \Gamma', \Delta') \rightsquigarrow_{suc}^{\theta'} (\Gamma', \Delta') \\ \wedge (\{G_1^*\}, \Gamma', \Delta') \not\rightsquigarrow_{ff} (\Gamma'', \Delta'') \\ \wedge (\{(t, G_1^*)\}, \Gamma', \Delta') \not\rightsquigarrow_{adj} (\Gamma'', \Delta'')]. \end{aligned}$$

It follows that

$$\forall \Gamma', \Delta' : [(\Gamma \subseteq \Gamma') \wedge (\Delta \subseteq \Delta') \Rightarrow (G^*, \Gamma', \Delta') \rightsquigarrow_{suc}^{\theta} (\Gamma', \Delta')].$$

Because G_1^* is involved in the derivation from G^* ,

$$\begin{aligned} \forall \Gamma', \Delta', \Gamma'', \Delta'' : [(\Gamma \subseteq \Gamma') \wedge (\Delta \subseteq \Delta') \Rightarrow \\ (\{G_1^*\}, \Gamma', \Delta') \not\rightsquigarrow_{ff} (\Gamma'', \Delta'') \wedge (\{(t, G_1^*)\}, \Gamma', \Delta') \not\rightsquigarrow_{adj} (\Gamma'', \Delta'')]. \end{aligned}$$

(ii) In case of (suc2): Consider the case that $L_s = A^*$ (ground), $A^* \in \Delta_0$, and $(G_1^*, \Gamma_0, \Delta_0) \rightsquigarrow_{suc} (\theta, \Gamma, \Delta)$ of length m , where

$$G_1^* = \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n.$$

By induction hypothesis,

$$\begin{aligned} \forall \Gamma', \Delta', \Gamma'', \Delta'' : [(\Gamma \subseteq \Gamma') \wedge (\Delta \subseteq \Delta') \Rightarrow \\ (G_1^*, \Gamma', \Delta') \rightsquigarrow_{suc}^{\theta} (\Gamma', \Delta') \\ \wedge (\{G_1^*\}, \Gamma', \Delta') \not\rightsquigarrow_{ff} (\Gamma'', \Delta'') \\ \wedge (\{(t, G_1^*)\}, \Gamma', \Delta') \not\rightsquigarrow_{adj} (\Gamma'', \Delta'')]. \end{aligned}$$

Because $A^* \in \Delta_0 \subseteq \Delta \subseteq \Delta'$, the induction step is completed.

(iii) In case of (suc3): Consider the case that $L_s = A^*$ (ground), $A^* \notin \Gamma_0 \cup \Delta_0$, $(\{\leftarrow A\}, \Gamma_0, \Delta_0 \cup \{A^*\}) \rightsquigarrow_{ff} (\Gamma'_0, \Delta'_0)$, and $(G_1^*, \Gamma'_0, \Delta'_0) \rightsquigarrow_{suc} (\theta, \Gamma, \Delta)$ of length m , where

$$G_1^* = \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n.$$

Because $A^* \in \Delta'_0 \subseteq \Delta \subseteq \Delta'$, the induction step is completed. q.e.d.

Proof of Lemma 4.6 It is proved by induction on the rank r of μ . Let $F = \{G_1^*, \dots, G_l^*\}$.

(1) In case that $r = 0$: It is proved by induction on the length k of the derivation for $(F, \Gamma_b, \Delta_b) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$.

(i) If $k = 0$, then $F = \emptyset$ and it holds.

(ii) Assume that it holds for $k = m$. Let $k = m + 1$. Also let $F = F' \cup \{\leftarrow L_1 \dots L_n\}$, where the selected literal in $\leftarrow L_1 \dots L_n$ is L_s .

(a) In case of (ff1): Assume that $L_s = A$, $\{g_1^*, \dots, g_p^*\}$ is a set of derived abductive goals by using A and

$$(F' \cup \{g_1^*, \dots, g_p^*\}, \Gamma_b, \Delta_b) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$$

of length m . By induction hypothesis,

$$\begin{aligned} \forall j, \Gamma', \Delta', \Gamma_z, \Delta_z, \theta_z : & [(1 \leq j \leq p) \Rightarrow \\ & [(\Gamma \subseteq \Gamma') \wedge (\Delta \subseteq \Delta' \subseteq \Delta'') \Rightarrow (g_j^*, \Gamma', \Delta') \not\rightsquigarrow_{suc}^{\theta_z} (\Gamma_z, \Delta_z)]]]. \end{aligned}$$

As well,

$$\begin{aligned} \forall i, \Gamma', \Delta', \Gamma_z, \Delta_z, \theta_z : & [(1 \leq i \leq l) \\ \Rightarrow & [(G_i^* \neq \leftarrow L_1 \dots L_n) \wedge (\Gamma \subseteq \Gamma') \wedge (\Delta \subseteq \Delta' \subseteq \Delta'') \\ \Rightarrow & (G_i^*, \Gamma', \Delta') \not\rightsquigarrow_{suc}^{\theta_z} (\Gamma_z, \Delta_z)]]]. \end{aligned}$$

It follows that

$$\begin{aligned} \forall i, \Gamma', \Delta', \Gamma_z, \Delta_z, \theta_z : & [(1 \leq i \leq l) \Rightarrow \\ & [(\Gamma \subseteq \Gamma') \wedge (\Delta \subseteq \Delta' \subseteq \Delta'') \Rightarrow \\ & (G_i^*, \Gamma', \Delta') \not\rightsquigarrow_{suc}^{\theta_z} (\Gamma_z, \Delta_z)]]]. \end{aligned}$$

This completes the induction step.

(b) In case of (ff2): Assume that $L_s = A^*$ (ground), $A^* \in \Gamma_b \cup \Delta_b$ and

$$(F' \cup \{\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n\}, \Gamma_b, \Delta_b) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$$

of length m . By induction hypothesis,

$$\begin{aligned} \forall i, \Gamma', \Delta', \Gamma_z, \Delta_z, \theta_z : & [(1 \leq i \leq l) \Rightarrow \\ & [(G_i^* \neq \leftarrow L_1 \dots L_n \vee G_i^* = \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n) \\ & \wedge (\Gamma \subseteq \Gamma') \wedge (\Delta \subseteq \Delta' \subseteq \Delta'') \Rightarrow (G_i^*, \Gamma', \Delta') \not\rightsquigarrow_{suc}^{\theta_z} (\Gamma_z, \Delta_z)]]]. \end{aligned}$$

(b-1) If $A^* \in \Delta_b$, then $A^* \in \Delta_b \subseteq \Delta \subseteq \Delta'$. Hence

$$\begin{aligned} \forall \Gamma', \Delta', \Gamma_z, \Delta_z, \theta_z : & \\ & [(\Gamma \subseteq \Gamma') \wedge (\Delta \subseteq \Delta' \subseteq \Delta'') \Rightarrow \\ & (\leftarrow L_1 \dots L_n, \Gamma', \Delta') \not\rightsquigarrow_{suc}^{\theta_z} (\Gamma_z, \Delta_z)]. \end{aligned}$$

This completes the induction step.

(b-2) If $A^* \notin \Delta_b$, then $A^* \in \Gamma_b$. Because $\Gamma_b \subseteq \Gamma \subseteq \Gamma'$, neither (suc1), (suc2) nor (suc3) can't be applied to $L_s = A^*$ in $\leftarrow L_1 \dots L_n$. This leads to that

$$\begin{aligned} \forall \Gamma', \Delta', \Gamma_z, \Delta_z, \theta_z : & [(\Gamma \subseteq \Gamma') \wedge (\Delta \subseteq \Delta' \subseteq \Delta'') \Rightarrow \\ & (\leftarrow L_1 \dots L_n, \Gamma', \Delta') \not\rightsquigarrow_{suc}^{\theta_z} (\Gamma_z, \Delta_z)]. \end{aligned}$$

This completes the induction step.

(c) The case of applying (ff3) is none, because the rank r is 0.

(2) Assume that it holds for $r \leq t$. Let $r = t + 1$. It is proved by induction on length k for $(F, \Gamma_b, \Delta_b) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$.

(i) If $k = 0$, it is proved like the case that $r = 0$.

(ii) Assume that it holds for $k = m$. Let $k = m + 1$. Also let $F = F' \cup \{\leftarrow L_1 \dots L_n\}$, where L_s is the selected literal in $\leftarrow L_1 \dots L_n$.

(a) In case of (ff1): It is proved like the case that $r = 0$.

(b) In case of (ff2): It is proved like the case that $r = 0$.

(c) In case of (ff3): Assume that $L_s = A^*$ and $A^* \notin \Gamma_b \cup \Delta_b$.

(c-1) In case of (ff3-1): Assume that $\beta' : (\leftarrow A, \Gamma_b, \Delta_b) \rightsquigarrow_{suc}^{\varepsilon} (\Gamma'_b, \Delta'_b)$ for some Γ'_b and Δ'_b such that $\mu > \beta'$, and

$$(F', \Gamma'_b, \Delta'_b) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$$

of length m . We prove that

$$\forall \Delta' : [(\Delta \subseteq \Delta' \subseteq \Delta'') \Rightarrow A^* \notin \Delta'].$$

Assume that $A^* \in \Delta$: Because $A^* \notin \Gamma_b \cup \Delta_b$,

$$\mu' : (\{\leftarrow A\}, \Gamma_d, \Delta_d) \rightsquigarrow_{ff} (\Gamma'_d, \Delta'_d) \text{ for some } \Gamma_d, \Delta_d, \Gamma'_d \text{ and } \Delta'_d \text{ and } \gamma \gg \mu'.$$

Because $A^* \in \Delta_d$, μ' occurs after β' starts. If $\beta' \gg \mu'$, then, by induction hypothesis for the rank (of μ') less than $t + 1$,

$$(\leftarrow A, \Gamma, \Delta) \not\rightsquigarrow_{suc}^{\theta_z} (\Gamma_z, \Delta_z).$$

Because $\Gamma'_b \subseteq \Gamma$ and $\Delta'_b \subseteq \Delta$, owing to Lemma 4.5, the induction hypothesis contradicts β' . If $\beta' \gg \mu'$, then $\Gamma'_b \subseteq \Gamma_d$ and $\Delta'_b \subseteq \Delta_d$. It contradicts μ' , by Lemma 4.5. Hence $A^* \notin \Delta$.

Assume that $A^* \in \Delta' - \Delta$. By the definition of Δ'' ,

$$\exists \Gamma_c, \Delta_c, \Gamma'_c, \Delta'_c : [(\Gamma \subseteq \Gamma_c) \wedge (\Delta \subseteq \Delta_c) \wedge (\{\leftarrow A\}, \Gamma_c, \Delta_c) \rightsquigarrow_{ff} (\Gamma'_c, \Delta'_c)].$$

Because $\Gamma'_b \subseteq \Gamma \subseteq \Gamma_c$ and $\Delta'_b \subseteq \Delta \subseteq \Delta_c$, it contradicts Lemma 4.5. Hence $A^* \notin \Delta' - \Delta$.

These case analyses lead us to that $A^* \notin \Delta'$. Next we conclude that

$$\forall \Gamma_z, \Delta_z, \theta_z : [(\leftarrow L_1 \dots L_n, \Gamma', \Delta') \not\rightsquigarrow_{suc}^{\theta_z} (\Gamma_z, \Delta_z)]$$

for the reasons:

Assume that $A^* \in \Gamma'$. Because $A^* \notin \Delta'$, (suc2) and (suc3) are not applied, it is evident.

Assume that $A^* \notin \Gamma'$. Because $A^* \notin \Delta'$, $A^* \notin \Gamma' \cup \Delta'$. Since $\Gamma'_b \subseteq \Gamma \subseteq \Gamma'$ and $\Delta'_b \subseteq \Delta \subseteq \Delta'$, by Lemma 4.5,

$$(\{\leftarrow A\}, \Gamma', \Delta' \cup \{A^*\}) \not\rightsquigarrow_{ff} (\Gamma'_z, \Delta'_z),$$

which makes it hold that $(\leftarrow L_1 \dots L_n, \Gamma', \Delta') \not\rightsquigarrow_{suc}^{\theta_z} (\Gamma_z, \Delta_z)$.

Combining that $\forall \Gamma_z, \Delta_z, \theta_z : [(\leftarrow L_1 \dots L_n, \Gamma', \Delta') \not\rightsquigarrow_{suc}^{\theta_z} (\Gamma_z, \Delta_z)]$ and $(F', \Gamma'_b, \Delta'_b) \not\rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$, we complete the induction step.

(c-2) In case of (ff3-2): Assume that $\mu' : (\{\leftarrow A\}, \Gamma_b, \Delta_b \cup \{A^*\}) \rightsquigarrow_{ff} (\Gamma'_b, \Delta'_b)$ for some Γ'_b and Δ'_b such that $\mu > \mu'$, and

$$(F' \cup \{\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n\}, \Gamma'_b, \Delta'_b) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$$

of length m . By induction hypothesis for the derivation of length m , and by that $A^* \in \Delta'_b \subseteq \Delta \subseteq \Delta'$,

$$\forall \Gamma', \Delta', \Gamma_z, \Delta_z, \theta_z : [(\Gamma \subseteq \Gamma') \wedge (\Delta \subseteq \Delta' \subseteq \Delta'') \Rightarrow (\leftarrow L_1 \dots L_n, \Gamma', \Delta') \not\sim_{suc}^{\theta_z} (\Gamma_z, \Delta_z)].$$

This can complete the induction step.

(c-3) In case of (ff3-3): Assume that $\nu' : (\{(\leftarrow A)\}, \Gamma_b \cup \{A^*\}, \Delta_b) \rightsquigarrow_{adj} (\Gamma'_b, \Delta'_b)$ for some Γ'_b and Δ'_b such that $\nu > \nu'$, and

$$(F' \cup \{\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n\}, \Gamma'_b, \Delta'_b) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$$

of length m . Take any Γ' and Δ' such that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta' \subseteq \Delta''$.

Assume that $A^* \in \Delta'$. By induction hypothesis for the derivation of length m ,

$$\forall \Gamma_z, \Delta_z, \theta_z : (\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n, \Gamma', \Delta') \not\sim_{suc}^{\theta_z} (\Gamma_z, \Delta_z).$$

Hence

$$\forall \Gamma_z, \Delta_z, \theta_z : (\leftarrow L_1 \dots L_n, \Gamma', \Delta') \not\sim_{suc}^{\theta_z} (\Gamma_z, \Delta_z).$$

This completes the induction step.

(c-4) Otherwise: Assume that

$$(F' \cup \{\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n\}, \Gamma_b, \Delta_b) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$$

of length m . For any Γ' and Δ' such that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta' \subseteq \Delta''$, in case that

$$(\leftarrow L_1 \dots L_n, \varepsilon, \Gamma', \Delta'), (\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n, \varepsilon, \Gamma', \Delta')$$

is a succeeding derivation. By applying induction hypothesis for the derivation of length m , we conclude that

$$(\leftarrow L_1 \dots L_n, \Gamma', \Delta') \not\sim_{suc}^{\theta_z} (\Gamma_z, \Delta_z),$$

and in case that $A^* \notin \Delta'$ and $A^* \in \Gamma'$, neither (suc1), (suc2) nor (suc3) is applied for the selected literal L_s in $(\leftarrow L_1 \dots L_n, \varepsilon, \Gamma', \Delta')$, and thus

$$(\leftarrow L_1 \dots L_n, \Gamma', \Delta') \not\sim_{suc}^{\theta_z} (\Gamma_z, \Delta_z).$$

Assume that $A^* \notin \Delta'$ and $A^* \in \Gamma'$. With the selected literal L_s in $\leftarrow L_1 \dots L_n$, neither (suc1), (suc2), nor (suc3) is applied. It follows that

$$\forall \Gamma_z, \Delta_z, \theta_z : (\leftarrow L_1 \dots L_n, \Gamma', \Delta') \not\sim_{suc}^{\theta_z} (\Gamma_z, \Delta_z).$$

This completes the induction step.

Assume that $A^* \notin \Delta'$, $A^* \notin \Gamma'$, and $\mu' : (\{\leftarrow A\}, \Gamma', \Delta' \cup \{A^*\}) \rightsquigarrow_{ff} (\Gamma_d, \Delta_d)$ for some Γ_d and Δ_d . It follows that

$$(\leftarrow L_1 \dots L_n, \varepsilon, \Gamma', \Delta'), (\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n, \varepsilon, \Gamma_d, \Delta_d)$$

is a succeeding derivation. We show below that $\Delta_d \subseteq \Delta''$. Assume that $a^* \in \Delta_d$ and $a^* \notin \Delta''$ for some a^* . $a^* \notin \Delta''$ implies $a^* \notin \Delta'$. As well, $a^* \in \Delta_d$ implies that

$$\mu'' : (\{\leftarrow a\}, \Gamma_e, \Delta_e) \rightsquigarrow_{ff} (\Gamma'_e, \Delta'_e) \text{ such that } \mu' \gg \mu'', \text{ or } a^* = A^*.$$

By the definition of Δ'' , $a^* \notin \Delta''$ implies

$$\forall \Gamma_c, \Delta_c, \Gamma'_c, \Delta'_c : [\Gamma \subseteq \Gamma_c \wedge \Delta \subseteq \Delta_c \Rightarrow (\{\leftarrow a\}, \Gamma_c, \Delta_c) \not\sim_{ff} (\Gamma'_c, \Delta'_c)].$$

If $a^* = A^*$, then this contradicts μ' , because $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta' \cup \{A^*\}$. If $a^* \neq A^*$, then it contradicts μ'' , because $\Gamma \subseteq \Gamma' \subseteq \Gamma_e$ and $\Delta \subseteq \Delta' \cup \{A^*\} \subseteq \Delta_e$. This concludes that $\Delta_d \subseteq \Delta''$. By induction hypothesis for

$$(F' \cup \{\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n\}, \Gamma_b, \Delta_b) \sim_{ff} (\Gamma'_a, \Delta'_a),$$

we see that

$$(\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n, \Gamma_d, \Delta_d) \not\sim_{suc}^{\theta_z} (\Gamma'_z, \Delta'_z),$$

because $\Gamma \subseteq \Gamma_d$ and $\Delta \subseteq \Delta_d \subseteq \Delta''$. It follows that

$$\forall \Gamma_z, \Delta_z, \theta_z : [(\leftarrow L_1 \dots L_n, \Gamma', \Delta') \not\sim_{suc}^{\theta_z} (\Gamma_z, \Delta_z)].$$

This completes the induction step.

Assume that $A^* \notin \Delta'$, $A^* \notin \Gamma'$ and

$$\forall \Gamma_d, \Delta_d : [(\{\leftarrow A\}, \Gamma', \Delta' \cup \{A^*\}) \not\sim_{ff} (\Gamma_d, \Delta_d)].$$

It is easy to see that

$$\forall \Gamma_z, \Delta_z, \theta_z : [(\leftarrow L_1 \dots L_n, \Gamma', \Delta') \not\sim_{suc}^{\theta_z} (\Gamma_z, \Delta_z)].$$

This completes induction step.

q.e.d.

Proof of Lemma 4.7 It is proved by induction on rank r of ν .

(1) In case that $r = 0$, it is proved by induction on length k of the derivation for $(C, \Gamma_b, \Delta_b) \sim_{adj} (\Gamma'_a, \Delta'_a)$ in ν .

(i) If $k = 0$, then $C = \{(\mathbf{u}, \Box)\}$ and it holds.

(ii) Assume that it holds for $k = m$. Let $k = m + 1$. Also assume that $C = C' \cup \{(V, \leftarrow L_1 \dots L_n)\}$, where L_s is the selected literal in $\leftarrow L_1 \dots L_n$.

(a) In case of (adj1): Assume that $L_s = A$ (an atom), the set of abductive goals derived by using A is $\{g_1^*, \dots, g_l^*\}$ and

$$(C' \cup \{(V, g_1^*), \dots, (V, g_l^*)\}, \Gamma_b, \Delta_b) \sim_{adj} (\Gamma'_a, \Delta'_a)$$

of length m . By induction hypothesis,

$$(C' \cup \{(V, g_1^*), \dots, (V, g_l^*)\}, \Gamma, \Delta) \sim_{adj} (\Gamma, \Delta).$$

Because

$$(C, \Gamma, \Delta), (C' \cup \{(V, g_1^*), \dots, (V, g_l^*)\}, \Gamma, \Delta)$$

is an adjusting derivation, we can complete the induction step.

(b) In case of (adj2): Assume that $L_s = A^*$ (a ground atom), $A^* \in \Gamma_b$, $A^* \notin \Delta_b$ and

$$(C' \cup \{(\mathbf{u}, \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n)\}, \Gamma_b, \Delta_b) \sim_{adj} (\Gamma'_a, \Delta'_a)$$

of length m . If $A^* \in \Gamma_b$, then neither (suc3), (ff3) nor (adj4) is applied to A^* . It follows that $A^* \notin \Delta$. Because $A^* \in \Gamma_b \subseteq \Gamma$,

$$(C, \Gamma, \Delta), (C' \cup \{(\mathbf{u}, \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n)\}, \Gamma, \Delta)$$

is an adjusting derivation. By means of induction hypothesis for the derivation of length m , we complete the induction step.

(c) In case of (adj3): Assume that $L_s = A^*$ (a ground atom), $A^* \in \Delta_b$ and

$$(C' \cup \{(V, \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n)\}, \Gamma_b, \Delta_b) \rightsquigarrow_{adj} (\Gamma'_a, \Delta'_a)$$

of length m . Because $A^* \in \Delta_b \subseteq \Delta$,

$$(C, \Gamma, \Delta), (C' \cup \{(V, \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n)\}, \Gamma, \Delta)$$

is an adjusting derivation. By making use of induction hypothesis for the derivation of length m , we complete the induction step.

(d) In case of (adj4): There is no possibility, because $r = 0$.

(2) Assume that it holds for rank $r \leq t$. Let $r = t + 1$. It is proved by induction on the length k for $(C, \Gamma_b, \Delta_b) \rightsquigarrow_{adj} (\Gamma'_a, \Delta'_a)$.

(i) In case that $k = 0$. It holds as for $r = 0$.

(ii) Assume that it holds for $k = m$. Also assume that $k = m + 1$ and $C = C' \cup \{(V, \leftarrow L_1 \dots L_n)\}$, where L_s is the selected literal in $\leftarrow L_1 \dots L_n$.

(a) In case of (adj1): It is proved like the case $r = 0$.

(b) In case of (adj2): It is proved like the case $r = 0$.

(c) In case of (adj3): It is proved like the case $r = 0$.

(d) In case of (adj4): Assume that $L_s = A^*$ (a ground atom) and $A^* \notin \Gamma_b \cup \Delta_b$.

(d-1) In case that $\nu' : (\{(t, \leftarrow A)\}, \Gamma_b \cup \{A^*\}, \Delta_b) \rightsquigarrow_{adj} (\Gamma'_b, \Delta'_b)$ for some Γ'_b and Δ'_b such that $\nu > \nu'$, and

$$(C' \cup \{(u, \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n)\}, \Gamma'_b, \Delta'_b) \rightsquigarrow_{adj} (\Gamma'_a, \Delta'_a)$$

of length m . Note that $A^* \in \Gamma'_b \subseteq \Gamma$. We show below that $A^* \notin \Delta$. Let $A^* \in \Delta$. Then

$$\mu' : (\{\leftarrow A\}, \Gamma_c, \Delta_c) \rightsquigarrow_{ff} (\Gamma'_c, \Delta'_c) \text{ for some } \Gamma_c, \Delta_c, \Gamma'_c \text{ and } \Delta'_c \text{ such that } \gamma \gg \mu'.$$

Because $A^* \notin \Delta_b$ and $A^* \in \Delta_c$, ν' does not appear after μ' . It follows that $A^* \in \Gamma'_b \subseteq \Gamma_c$. This contradicts the application of (suc3), (ff3) or (adj4) for A^* . It is impossible that $\gamma \gg \mu'$. Hence $A^* \notin \Delta$. By applying (adj2), we have an adjusting derivation:

$$(C, \Gamma, \Delta), (C' \cup \{(u, \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n)\}, \Gamma, \Delta).$$

By making use of induction hypothesis for the derivation of length m , we complete the induction step.

(d-2) In case that $\beta' : (\leftarrow A, \Gamma_b, \Delta_b) \rightsquigarrow_{suc}^e (\Gamma'_b, \Delta'_b)$ for some Γ'_b and Δ'_b such that $\nu > \beta'$, and

$$(C', \Gamma'_b, \Delta'_b) \rightsquigarrow_{adj} (\Gamma'_a, \Delta'_a)$$

of length m . It is sufficient that $A^* \notin \Gamma \cup \Delta$ and $(\leftarrow A, \Gamma, \Delta) \rightsquigarrow_{suc}^e (\Gamma, \Delta)$, to see that

$$(C, \Gamma, \Delta), (C', \Gamma, \Delta)$$

is an adjusting derivation.

Assume that $A^* \in \Gamma$. Then

$$\nu' : (\{(t, \leftarrow A)\}, \Gamma_c, \Delta_c) \rightsquigarrow_{adj} (\Gamma'_c, \Delta'_c)$$

for some $\Gamma_c, \Gamma'_c, \Delta_c, \Delta'_c$ such that $\gamma \gg \nu'$. Because $A^* \notin \Gamma_b$ and $A^* \in \Gamma_c$, β' does not appear after ν' . If $\beta \gg \nu'$, then the rank of ν' is not greater than t . By induction hypothesis on rank,

$$(\{(t, \leftarrow A)\}, \Gamma, \Delta) \rightsquigarrow_{adj} (\Gamma, \Delta).$$

Because $\Gamma'_b \subseteq \Gamma$ and $\Delta'_b \subseteq \Delta$, this contradicts Lemma 4.5. Unless $\beta' \gg \nu'$, $\Gamma'_b \subseteq \Gamma_c$ and $\Delta'_b \subseteq \Delta_c$, which shows that ν' contradicts Lemma 4.5. Hence $A^* \notin \Gamma$.

Assume that $A^* \in \Delta$. Then

$$\mu' : (\{\leftarrow A\}, \Gamma_c, \Delta_c) \rightsquigarrow_{ff} (\Gamma'_c, \Delta'_c)$$

for some $\Gamma_c, \Delta_c, \Gamma'_c$ and Δ'_c such that $\gamma \gg \mu'$. Because $A^* \notin \Delta_b$ and $A^* \in \Delta_c$, β' does not appear after μ' . If $\beta' \gg \mu'$, then by Lemma 4.6,

$$\forall \Gamma_z, \Delta_z, \theta_z : [(\leftarrow A, \Gamma, \Delta) \not\rightsquigarrow_{suc}^{\theta_z} (\Gamma_z, \Delta_z)].$$

Because $\Gamma'_b \subseteq \Gamma$ and $\Delta'_b \subseteq \Delta$, this contradicts Lemma 4.5. Unless $\beta' \gg \mu'$, μ' contradicts Lemma 4.5, because $\Gamma'_b \subseteq \Gamma_c$ and $\Delta'_b \subseteq \Delta_c$. Hence $A^* \notin \Delta$.

Because $\Gamma'_b \subseteq \Gamma$ and $\Delta'_b \subseteq \Delta$, by Lemma 4.5,

$$(\leftarrow A, \Gamma, \Delta) \rightsquigarrow_{suc}^{\varepsilon} (\Gamma, \Delta).$$

By the adjusting derivation (C, Γ, Δ) , (C', Γ, Δ) , and by induction hypothesis for the derivation of length m , we complete the induction step for $(C, \Gamma, \Delta) \rightsquigarrow_{adj} (\Gamma, \Delta)$.

(d-3) In case that $\mu' : (\{\leftarrow A\}, \Gamma_b, \Delta_b \cup \{A^*\}) \rightsquigarrow_{ff} (\Gamma'_b, \Delta'_b)$ for some Γ'_b and Δ'_b such that $\nu > \mu'$, and

$$(C' \cup \{(V, \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n)\}, \Gamma'_b, \Delta'_b) \rightsquigarrow_{adj} (\Gamma'_a, \Delta'_a)$$

of length m . Because $A^* \in \Delta'_b \subseteq \Delta$,

$$(C, \Gamma, \Delta), (C' \cup \{(V, \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n)\}, \Gamma, \Delta)$$

is an adjusting derivation. By making use of induction hypothesis for the derivation of length m , we complete the induction step. q.e.d.

Proof of Lemma 4.8 Assume $\beta : (\leftarrow a, \Gamma_0, \Delta_0) \rightsquigarrow_{suc}^{\varepsilon} (\Gamma, \Delta)$.

(1) On the assumption that $a^* \notin \Gamma_0$ and $a^* \in \Gamma$,

$$\nu : (\{(t, \leftarrow a)\}, \Gamma_a, \Delta_a) \rightsquigarrow_{adj} (\Gamma'_a, \Delta'_a)$$

for some $\Gamma_a, \Delta_a, \Gamma'_a, \Delta'_a$ such that $\beta \gg \nu$. By Lemma 4.7,

$$(\{(t, \leftarrow a)\}, \Gamma, \Delta) \rightsquigarrow_{adj} (\Gamma, \Delta).$$

This contradicts Lemma 4.5. Hence $a^* \notin \Gamma_0$ implies $a^* \notin \Gamma$.

(2) On the assumption that $a^* \notin \Delta_0$ and $a^* \in \Delta$,

$$\mu : (\{\leftarrow a\}, \Gamma_a, \Delta_a) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$$

for some $\Gamma_a, \Delta_a, \Gamma'_a$ and Δ'_a such that $\beta \gg \mu$. By Lemma 4.6,

$$(\leftarrow a, \Gamma, \Delta) \not\rightsquigarrow_{suc}^{\theta_z} (\Gamma_z, \Delta_z)$$

for any Γ_z , Δ_z and θ_z . This contradicts Lemma 4.5. Hence $a^* \notin \Delta_0$ implies $a^* \notin \Delta$. q.e.d.

Proof of Lemma 4.9 It is proved by induction on the length k for $(F, \Gamma_b, \Delta_b) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$.

(1) If $k = 0$, then $F = \emptyset$ and it holds.

(2) Assume that it holds for $k \leq m$. Let $k = m + 1$ and $F = F' \cup \{\leftarrow L_1 \dots L_n\}$, where L_s is the selected literal in $\leftarrow L_1 \dots L_n$.

(i) In case of (ff1): Assume that $L_s = A$, the set of abductive goals is $\{g_1^*, \dots, g_p^*\}$ derived by using A , and

$$(F' \cup \{g_1^*, \dots, g_p^*\}, \Gamma_b, \Delta_b) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$$

of length m .

(a) If $p = 0$, then there is no abductive goal by using A , and $L_s \varphi \in B_P$ implies $L_s \varphi \notin S_P(\Gamma^+ \cup \Delta^+)$.

(b) On the assumption that $p > 0$, we assume

$$\begin{aligned} \exists \varphi(\text{ground substitution}), \forall j : & [(1 \leq j \leq n) \Rightarrow \\ & [(L_j \varphi = a \in B_P \wedge a \in S_P(\Gamma^+ \cup \Delta^+)) \\ & \vee (L_j \varphi = a^* \in (B_P)^* \wedge a^* \in \Gamma \cup \Delta)]] \quad (\#) \end{aligned}$$

Also let $g_i^* = \leftarrow (L_1 \dots L_{s-1} l_1^i \dots l_{p_i}^i L_{s+1} \dots L_n) \theta_i$ for $A_i \leftarrow l_1^i \dots l_{p_i}^i \in P^*$ and $\theta_i = \text{mgu}(A, A_i)$.

In case that $\varphi = \theta_i \rho$ (a ground substitution) for some i ($1 \leq i \leq p$) and ρ , $A \varphi = A \theta_i \rho = A_i \theta_i \rho$. By the assumption (#), $A \varphi \in S_P(\Gamma^+ \cup \Delta^+)$. That is, $A_i \theta_i \rho \in S_P(\Gamma^+ \cup \Delta^+)$. Hence

$$\begin{aligned} \exists (A_i \leftarrow l_1^i \dots l_{p_i}^i) \theta_i \rho \xi \in \text{ground}(P^*), \forall j : & \\ [(1 \leq j \leq p_i) \Rightarrow & \\ [(l_j^i \theta_i \rho \xi = a \in B_P \Rightarrow a \in S_P(\Gamma^+ \cup \Delta^+)) & \\ \wedge (l_j^i \theta_i \rho \xi = a^* \in (B_P)^* \Rightarrow a^* \in \Gamma \cup \Delta)]] \quad (*) \end{aligned}$$

By induction hypothesis for the derivation of length m , applied to g_i^* , for any ground substitution σ ,

$$\begin{aligned} \text{(b-1)} \quad \exists j : & [(1 \leq j \leq n, j \neq s) \wedge \\ & [(L_j \theta_i \sigma = a \in B_P \Rightarrow a \notin S_P(\Gamma^+ \cup \Delta^+)) \\ & \wedge (L_j \theta_i \sigma = a^* \in (B_P)^* \Rightarrow a^* \notin \Gamma \cup \Delta)]] \text{ or} \\ \text{(b-2)} \quad \exists j : & [(1 \leq j \leq p_i) \wedge \\ & [(l_j^i \theta_i \sigma = a \in B_P \Rightarrow a \notin S_P(\Gamma^+ \cup \Delta^+)) \\ & \wedge (l_j^i \theta_i \sigma = a^* \in (B_P)^* \Rightarrow a^* \notin \Gamma \cup \Delta)]] \end{aligned}$$

The case (b-1) contradicts (#), while the case (b-2) contradicts (*).

In case that there is no i such that $\varphi = \theta_i \rho$ for any ρ , there is no clause in P^* , whose head is unifiable to $A \varphi$. It follows that $A \varphi \notin S_P(\Gamma^+ \cup \Delta^+)$, which contradicts (#).

Therefore

$$\begin{aligned} \forall \varphi, \exists j : & [(1 \leq j \leq n) \wedge \\ & [(L_j \varphi = a \in B_P \Rightarrow a \notin S_P(\Gamma^+ \cup \Delta^+)) \wedge (L_j \varphi = a^* \in (B_P)^* \Rightarrow a^* \notin \Gamma \cup \Delta)]] \end{aligned}$$

By making use of induction hypothesis for the derivation of length m , we complete the induction step.

(ii) In case of (ff2): Assume that $L_s = A^*$ (a ground atom), $A^* \in \Gamma_b \cup \Delta_b$, and

$$(F' \cup \{\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n\}, \Gamma_b, \Delta_b) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$$

of length m . By induction hypothesis for the derivation of length m , the induction is completed.

(iii) In case of (ff3): Assume that $L_s = A^*$ (a ground atom) and $A^* \notin \Gamma_b \cup \Delta_b$.

(a) In case that $\beta' : (\leftarrow A, \Gamma_b, \Delta_b) \rightsquigarrow_{suc}^{\varepsilon} (\Gamma'_b, \Delta'_b)$ and

$$(F', \Gamma'_b, \Delta'_b) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$$

of length m . Since induction hypothesis can be applied to the derivation of length m , it is sufficient to show for the completion of the induction step that $A^* \notin \Gamma \cup \Delta$. Assume that $A^* \in \Gamma \cup \Delta$. By Lemma 4.8, $A^* \notin \Gamma'_b \cup \Delta'_b$. It follows that

$$\begin{aligned} \text{(a-1)} \quad & \nu' : (\{(t, \leftarrow A)\}, \Gamma_c, \Delta_c) \rightsquigarrow_{adj} (\Gamma'_c, \Delta'_c) \text{ such that } \gamma \gg \nu', \text{ or} \\ \text{(a-2)} \quad & \mu' : (\{\leftarrow A\}, \Gamma_c, \Delta_c) \rightsquigarrow_{ff} (\Gamma'_c, \Delta'_c) \text{ such that } \gamma \gg \mu'. \end{aligned}$$

In case of (a-1), $A^* \in \Gamma_c$ and β' does not appear after ν' . Hence $\Gamma'_b \subseteq \Gamma_c$ and $\Delta'_b \subseteq \Delta_c$. This contradicts Lemma 4.5. In case of (a-2), $A^* \in \Delta_c$ and β' does not appear after μ' . Hence $\Gamma'_b \subseteq \Gamma_c$ and $\Delta'_b \subseteq \Delta_c$. This contradicts Lemma 4.5. Therefore $A^* \notin \Gamma \cup \Delta$.

(b) In case that $(\leftarrow A, \Gamma_b, \Delta_b \cup \{A^*\}) \rightsquigarrow_{ff} (\Gamma'_b, \Delta'_b)$ and

$$(F' \cup \{\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n\}, \Gamma'_b, \Delta'_b) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$$

of length m , it is easy to see that the induction step is completed by applying induction hypothesis to the derivation of length m .

(c) In case that $(\{(t, \leftarrow A)\}, \Gamma_b \cup \{A^*\}, \Delta_b) \rightsquigarrow_{adj} (\Gamma'_b, \Delta'_b)$ and

$$(F' \cup \{\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n\}, \Gamma'_b, \Delta'_b) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$$

of length m , it is easy that the induction step is completed by applying induction hypothesis to the derivation of length m .

(d) Assume that

$$(F' \cup \{\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n\}, \Gamma_b, \Delta_b) \rightsquigarrow_{ff} (\Gamma'_a, \Delta'_a)$$

of length m . By applying induction hypothesis to this derivation, we complete the induction step. q.e.d.

Proof of Lemma 4.10 It is proved by induction on the length k for $(F, \Gamma_0, \Delta_0) \rightsquigarrow_{ff} (\Gamma, \Delta)$.

(1) If $k = 0$, then $l = 0$ and it holds.

(2) Assume that it holds for $k = m$. Let $k = m + 1$. Assume that $F = F' \cup \{\leftarrow L_1 \dots L_n\}$, where L_s is the selected literal in $\leftarrow L_1 \dots L_n$.

(i) In case of (ff1): Assume that $L_s = A$, the set of abductive goals derived by using A is $\{g_1^*, \dots, g_p^*\}$, and

$$(F' \cup \{g_1^*, \dots, g_p^*\}, \Gamma_0, \Delta_0) \rightsquigarrow_{ff} (\Gamma, \Delta)$$

of length m .

(a) If $p = 0$, then there is no derived abductive goal by using A , and $L_s \varphi \in B_P$ implies $L_s \varphi \notin S_P(\overline{S_P(\Delta^+)})$.

(b) Assume that $p > 0$. Also assume that

$$\begin{aligned} & \exists \varphi(\text{ground substitution}), \forall j : [(1 \leq j \leq n) \Rightarrow \\ & [(L_j \varphi = a \in B_P \vee a \in S_P(\overline{S_P(\Delta^+)})] \vee (L_j \varphi = a^* \in (B_P)^* \wedge a \notin S_P(\Delta^+)))] \quad (\#) \end{aligned}$$

Let $g_i^* \leftarrow (L_1 \dots L_{s-1} l_1^i \dots L_{p_i}^i L_{s+1} \dots L_n) \theta_i$ for $A_i \leftarrow l_1^i \dots l_{p_i}^i \in P^*$ such that $\theta_i = \text{mgu}(A, A_i)$ ($1 \leq i \leq p$).

In case that $\varphi = \theta_i \rho$ (a ground substitution) for some i , $A\varphi = A\theta_i \rho = A_i \theta_i \rho$. By (#), $A\varphi \in S_P(\overline{S_P(\Delta^+)})$, that is, $A_i \theta_i \rho \in S_P(\overline{S_P(\Delta^+)})$. Hence

$$\begin{aligned} & \exists (A_i \leftarrow l_1^i \dots l_{p_i}^i) \theta_i \rho \xi \in \text{ground}(P^*), \forall j : [(1 \leq j \leq p_i) \Rightarrow \\ & [(l_j^i \theta_i \rho \xi = a \in B_P \Rightarrow a \in S_P(\overline{S_P(\Delta^+)})] \wedge (l_j^i \theta_i \rho \xi = a^* \in (B_P)^* \Rightarrow a \notin S_P(\Delta^+))]] \quad (*) \end{aligned}$$

By induction hypothesis for g_i^* , for any ground substitution σ ,

$$\begin{aligned} \text{(b-1)} \quad \exists j : & [(1 \leq j \leq n) \wedge (j \neq s) \wedge \\ & [(L_j \theta_i \sigma = a \in B_P \Rightarrow a \notin S_P(\overline{S_P(\Delta^+)})] \\ & \wedge (L_j \theta_i \sigma = a^* \in (B_P)^* \Rightarrow a \in S_P(\Delta^+))]] \end{aligned}$$

or

$$\begin{aligned} \text{(b-2)} \quad \exists j : & [(1 \leq j \leq p_i) \wedge \\ & [(l_j^i \theta_i \sigma = a \in B_P \Rightarrow a \notin S_P(\overline{S_P(\Delta^+)})] \\ & \wedge (l_j^i \theta_i \sigma = a^* \in (B_P)^* \Rightarrow a \in S_P(\Delta^+))]] \end{aligned}$$

The case (b-1) contradicts (#), because $\varphi = \theta_i \rho$, while the case (b-2) contradicts (*).

In case that there is no i such that $\varphi = \theta_i \rho$, there is no clause in P^* , whose head is unifiable with $A\varphi$. Hence $A\varphi \notin S_P(\overline{S_P(\Delta^+)})$, which contradicts (#).

It follows that

$$\begin{aligned} & \forall \varphi (\text{ground substitution}), \exists j : [(1 \leq j \leq n) \wedge \\ & [(L_j \varphi = a \in B_P \Rightarrow a \notin S_P(\overline{S_P(\Delta^+)})] \wedge (L_j \varphi = a^* \in (B_P)^* \Rightarrow a \in S_P(\Delta^+))]]. \end{aligned}$$

By making use of induction hypothesis for the derivation of length m , we complete the induction step.

(ii) In case of (ff2): Assume that $L_s = A^*$ (a ground atom), $A^* \in \Gamma_0 \cup \Delta_0$ and

$$(F' \cup \{\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n\}, \Gamma_0, \Delta_0) \rightsquigarrow_{ff} (\Gamma, \Delta)$$

of length m . By induction hypothesis for the derivation of length m , we complete the induction step.

(iii) In case of (ff3): Assume that $L_s = A^*$ (a ground atom) and $A^* \notin \Gamma_0 \cup \Delta_0$.

(a) In case of (ff3-1): In case that $(\leftarrow A, \Gamma_0, \Delta_0) \rightsquigarrow_{suc}^{\epsilon} (\Gamma'_0, \Delta'_0)$ and

$$(F', \Gamma'_0, \Delta'_0) \rightsquigarrow_{ff} (\Gamma, \Delta)$$

of length m , it is sufficient for the completion of induction step to show that $A \in S_P(\Delta^+)$. By Lemma 4.4, $A \in S_P(\Delta_0'^+)$. Because $\Delta_0' \subseteq \Delta$ and S_P is monotonic, $A \in S_P(\Delta^+)$.

(b) In case of (ff3-2): In case that $(\{\leftarrow A\}, \Gamma_0, \Delta_0 \cup \{A^*\}) \rightsquigarrow_{ff} (\Gamma'_0, \Delta'_0)$,

$$(F' \cup \{\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n\}, \Gamma'_0, \Delta'_0) \rightsquigarrow_{ff} (\Gamma, \Delta)$$

of length m . By applying induction hypothesis for this derivation, we complete the induction step.

(c) In case of (ff3-3): In case that $(\{\leftarrow A\}, \Gamma_0 \cup \{A^*\}, \Delta_0) \rightsquigarrow_{adj} (\Gamma'_0, \Delta'_0)$ and

$$(F' \cup \{\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n\}, \Gamma'_0, \Delta'_0) \rightsquigarrow_{ff} (\Gamma, \Delta)$$

of length m . By applying induction hypothesis for this derivation, we complete the induction step.

(d) In case of (ff3-4): Otherwise,

$$(F' \cup \{\leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n\}, \Gamma_0, \Delta_0) \rightsquigarrow_{ff} (\Gamma, \Delta)$$

of length m . By applying induction hypothesis for this derivation, we complete the induction step. q.e.d.

Proof of Lemma 4.11 It is proved by induction on the length k of $\nu : (C, \Gamma_0, \Delta_0) \rightsquigarrow_{adj} (\Gamma, \Delta)$.

(1) If $k = 0$, then $C = \{(\mathbf{u}, \square)\}$ and it holds.

(2) Assume that it holds for $k = m$. Let $k = m + 1$. Also assume that $C = C' \cup \{(V, \leftarrow L_1 \dots L_n)\}$, where L_s is the selected literal in $\leftarrow L_1 \dots L_n$.

(i) In case of (adj1): Assume that $L_s = A$, the set of goals derived by using A is $\{g_1^*, \dots, g_p^*\}$, and

$$(C' \cup \{(V, g_1^*), \dots, (V, g_p^*)\}, \Gamma_0, \Delta_0) \rightsquigarrow_{adj} (\Gamma, \Delta)$$

of length m .

(a) If $p = 0$, the induction step is completed by applying induction hypothesis for the derivation of length m .

(b) In case that $p > 0$: Assume that the induction step is not completed by applying induction hypothesis for C' . Let

$$g_i^* = \leftarrow (L_1 \dots L_{s-1} l_1^{i_1} \dots l_{p_i}^{i_{p_i}} L_{s+1} \dots L_n) \theta_i,$$

where $A_i \leftarrow l_1^{i_1} \dots l_{p_i}^{i_{p_i}} \in P^*$ and $\theta_i = mgu(A, A_i)$ ($1 \leq i \leq p$). By induction hypothesis,

$$\begin{aligned} & \exists i, \varphi(\text{ground substitution}), \forall j : [(1 \leq i \leq p) \wedge \\ & [(1 \leq j \leq n) \wedge (j \neq s) \Rightarrow \\ & [(L_j \theta_i \varphi = a \in B_P \Rightarrow a \in S_P(\Gamma^+ \cup \Delta^+)) \wedge (L_j \theta_i \varphi = a^* \in (B_P)^* \Rightarrow a^* \in \Gamma \cup \Delta)]] \\ & \wedge [(1 \leq j \leq p_i) \Rightarrow \\ & [l_j^{i_j} \theta_i \varphi = a \in B_P \Rightarrow a \in S_P(\Gamma^+ \cup \Delta^+) \wedge l_j^{i_j} \theta_i \varphi = a^* \in (B_P)^* \Rightarrow a^* \in \Gamma \cup \Delta]]]. \end{aligned}$$

Hence

$$\exists \varphi(\text{ground substitution}) : [(1 \leq i \leq p) \wedge (A_i \theta_i \varphi \sigma = a \in B_P \Rightarrow a \in S_P(\Gamma^+ \cup \Delta^+))].$$

Because $A_i \theta_i = A \theta_i$,

$$A \theta_i \varphi \sigma = a \in B_P \Rightarrow a \in S_P(\Gamma^+ \cup \Delta^+).$$

This completes the induction step.

(ii) In case of (adj2): Assume that $L_s = A^*$ (a ground atom), $A^* \in \Gamma_0$, $A^* \notin \Delta_0$ and

$$(C' \cup \{(\mathbf{u}, \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n)\}, \Gamma_0, \Delta_0) \rightsquigarrow_{adj} (\Gamma, \Delta)$$

of length m . Assume that the induction step is not completed even by applying induction hypothesis for C' . By induction hypothesis for the derivation of length m ,

$$\begin{aligned} & \exists \varphi(\text{ground substitution}), \forall j : [(1 \leq j \leq n) \wedge (j \neq s) \Rightarrow \\ & [L_j \varphi = a \in B_P \Rightarrow a \in S_P(\Gamma^+ \cup \Delta^+) \wedge L_j \varphi = a^* \in (B_P)^* \Rightarrow a^* \in \Gamma \cup \Delta]]. \end{aligned}$$

Because $A^* \in \Gamma_0 \subseteq \Gamma$, the induction step is completed.

(iii) In case of (adj3): Assume that $L_s = A^*$ (a ground atom), $A^* \in \Delta_0$ and

$$(C' \cup \{(V, \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n)\}, \Gamma_0, \Delta_0) \rightsquigarrow_{adj} (\Gamma, \Delta).$$

Because $A^* \in \Delta_0 \subseteq \Delta$, the induction step is completed.

(iv) In case of (adj4): Assume that $L_s = A^*$ (a ground atom) and $A^* \notin \Gamma_0 \cup \Delta_0$.

(a) In case of (adj4-1): Assume that $(\{(\mathbf{t}, \leftarrow A)\}, \Gamma_0 \cup \{A^*\}, \Delta_0) \rightsquigarrow_{adj} (\Gamma'_0, \Delta'_0)$ and

$$(C' \cup \{(\mathbf{u}, \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n)\}, \Gamma'_0, \Delta'_0) \rightsquigarrow_{adj} (\Gamma, \Delta)$$

of length m . Because $A^* \in \Gamma'_0 \subseteq \Gamma$, the induction step is completed.

(b) In case of (adj4-2): Assume that $(\leftarrow A, \Gamma_0, \Delta_0) \rightsquigarrow_{suc}^{\varepsilon} (\Gamma'_0, \Delta'_0)$, and

$$(C', \Gamma'_0, \Delta'_0) \rightsquigarrow_{adj} (\Gamma, \Delta)$$

of length m . By applying induction hypothesis to the derivation of length m , the induction step is completed.

(c) In case of (adj4-3): Assume that $(\{\leftarrow A\}, \Gamma_0, \Delta_0 \cup \{A^*\}) \rightsquigarrow_{ff} (\Gamma'_0, \Delta'_0)$ and

$$(C' \cup \{(V, \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n)\}, \Gamma'_0, \Delta'_0) \rightsquigarrow_{adj} (\Gamma, \Delta).$$

Because $A^* \in \Delta'_0 \subseteq \Delta$, the induction step is completed. q.e.d.

Proof of Lemma 4.12 It is proved by induction on the length k for $\nu : (C, \Gamma_b, \Delta_b) \rightsquigarrow_{adj} (\Gamma'_a, \Delta'_a)$.

(1) If $k = 0$, then $C = \{(\mathbf{u}, \square)\}$ and it holds.

(2) Assume that it holds for $k = m$. Let $k = m + 1$. Also assume that $C = C' \cup \{(V, \leftarrow L_1 \dots L_n)\}$, where L_s is the selected literal in $\leftarrow L_1 \dots L_n$.

(i) In case of (adj1): Assume that $L_s = A$, the set of abductive goals derived by using A is $\{g_1^*, \dots, g_p^*\}$, and

$$(C' \cup \{(V, g_1^*), \dots, (V, g_p^*)\}, \Gamma_b, \Delta_b) \rightsquigarrow_{adj} (\Gamma'_a, \Delta'_a)$$

of length m . If $V = \mathbf{u}$, then the induction step is completed by applying induction hypothesis for C' . Let $V = \mathbf{t}$.

(a) If $p = 0$, then there is no derived abductive goal by using A , and $L_s \varphi \in B_P$ implies $L_s \notin S_P(\overline{S_P(\Gamma_0^+ \cup \Delta_0^+)})$. By applying induction hypothesis, the induction step is completed.

(b) Assume that $p > 0$. Also assume that

$$\begin{aligned} \exists \varphi, \forall j : [(1 \leq j \leq n) \Rightarrow \\ [L_j \varphi = a \in B_P \wedge a \in S_P(\overline{S_P(\Gamma_0^+ \cup \Delta_0^+)}) \text{ or} \\ L_j \varphi = a^* \in (B_P)^* \wedge a \notin S_P(\Gamma_0^+ \cup \Delta_0^+)]]. \quad (\#) \end{aligned}$$

Let

$$g_i^* = \leftarrow (L_1 \dots L_{s-1} l_1^i \dots l_{p_i}^i L_{s+1} \dots L_n) \theta_i$$

for $A_i \leftarrow l_1^i \dots l_{p_i}^i \in P^*$ such that $\theta_i = \text{mgu}(A, A_i)$.

In case that $\varphi = \theta_i \rho$ (a ground substitution) for some i , $A \varphi = A \theta_i \rho = A_i \theta_i \rho$. By $(\#)$, $A \varphi \in S_P(\overline{S_P(\Gamma_0^+ \cup \Delta_0^+)})$. That is, $A_i \theta_i \rho \in S_P(\overline{S_P(\Gamma_0^+ \cup \Delta_0^+)})$. Hence

$$\begin{aligned} \exists (A_i \leftarrow l_1^i \dots l_{p_i}^i) \theta_i \rho \xi \in \text{ground}(P^*), \forall j : [(1 \leq j \leq p_i) \Rightarrow \\ [l_j^i \theta_i \rho \xi = a \in B_P \Rightarrow a \in S_P(\overline{S_P(\Gamma_0^+ \cup \Delta_0^+)}) \wedge \\ l_j^i \theta_i \rho \xi = a^* \in (B_P)^* \Rightarrow a \notin S_P(\Gamma_0^+ \cup \Delta_0^+)]]. \quad (*) \end{aligned}$$

By induction hypothesis, for any ground substitution σ ,

$$\begin{aligned} & \exists j : [(1 \leq j \leq n) \wedge (j \neq s) \wedge \\ & \quad [L_j \theta_i \sigma = a \in B_P \Rightarrow a \notin S_P(\overline{S_P(\Gamma_0^+ \cup \Delta_0^+)})] \wedge \\ & \quad L_j \theta_i \sigma = a^* \in (B_P)^* \Rightarrow a \in S_P(\Gamma_0^+ \cup \Delta_0^+)] \text{ or} \\ & \exists j : [(1 \leq j \leq p_i) \wedge \\ & \quad [l_j^i \theta_i \sigma = a \in B_P \Rightarrow a \notin S_P(\overline{S_P(\Gamma_0^+ \cup \Delta_0^+)})] \wedge \\ & \quad l_j^i \theta_i \sigma = a^* \in (B_P)^* \Rightarrow a \in S_P(\Gamma_0^+ \cup \Delta_0^+)]]. \end{aligned}$$

The former case contradicts (#), because $\varphi = \theta_i \rho$. The latter case contradicts (*).

In case that $\varphi \neq \theta_i \rho$ for any i , there is no clause of P^* , whose head is unifiable with $A\varphi$. Hence $A\varphi \notin S_P(\overline{S_P(\Gamma_0^+ \cup \Delta_0^+)})$, which contradicts (#).

Therefore

$$\begin{aligned} & \forall \varphi, \exists j : [(1 \leq j \leq n) \wedge \\ & \quad [(L_j \varphi = a \in B_P \Rightarrow a \notin S_P(\overline{S_P(\Gamma_0^+ \cup \Delta_0^+)})] \\ & \quad \wedge (L_j \varphi = a^* \in (B_P)^* \Rightarrow a \in S_P(\Gamma_0^+ \cup \Delta_0^+))]. \end{aligned}$$

This is applied to the completion of induction step.

(ii) In case of (adj2): Assume that $L_s = A^*$ (a ground atom), $A^* \in \Gamma_b$, $A^* \notin \Delta_b$ and

$$(C' \cup \{(\mathbf{u}, \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n)\}, \Gamma_b, \Delta_b) \rightsquigarrow_{adj} (\Gamma'_a, \Delta'_a)$$

of length m . If $V = \mathbf{u}$, then the induction step is completed by applying induction hypothesis for the derivation of length m . Assume that $V = \mathbf{t}$. It is sufficient for the completion of the induction step to show that $A \in S_P(\Gamma_0^+ \cup \Delta_0^+)$. Because $A^* \in \Gamma_b$,

$$\nu' : (\{(\mathbf{t}, \leftarrow A)\}, \Gamma_c, \Delta_c) \rightsquigarrow_{adj} (\Gamma'_c, \Delta'_c) \text{ such that } \gamma \gg \nu'.$$

By Lemma 4.11, $A \in S_P(\Gamma_c^+ \cup \Delta_c^+)$. Since $\Gamma'_c \subseteq \Gamma_0$, $\Delta'_c \subseteq \Delta_0$ and S_P is monotonic, $A \in S_P(\Gamma_0^+ \cup \Delta_0^+)$.

(iii) In case of (adj3): Assume that $L_s = A^*$ (a ground atom), $A^* \in \Delta_b$ and

$$(C' \cup \{(V, \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n)\}, \Gamma_b, \Delta_b) \rightsquigarrow_{adj} (\Gamma'_a, \Delta'_a)$$

of length m . By applying induction hypothesis to this derivation, we complete the induction step.

(iv) In case of (adj4): Assume that $L_s = A^*$ (a ground atom) and $A^* \notin \Gamma_b \cup \Delta_b$.

(a) Assume that $(\{(\mathbf{t}, \leftarrow A)\}, \Gamma_b \cup \{A^*\}, \Delta_b) \rightsquigarrow_{adj} (\Gamma'_b, \Delta'_b)$ and

$$(C' \cup \{(\mathbf{u}, \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n)\}, \Gamma'_b, \Delta'_b) \rightsquigarrow_{adj} (\Gamma'_a, \Delta'_a)$$

of length m . If $V = \mathbf{u}$, then the induction step is completed by applying induction hypothesis. Assume that $V = \mathbf{t}$. It is sufficient for the completion of the induction step to show that $A \in S_P(\Gamma_0^+ \cup \Delta_0^+)$. By lemma 4.11, $A \in S_P(\Gamma_b^+ \cup \Delta_b^+)$. Because $\Gamma'_b \subseteq \Gamma_0$, $\Delta'_b \subseteq \Delta_0$ and S_P is monotonic, $A \in S_P(\Gamma_0^+ \cup \Delta_0^+)$.

(b) Assume that $(\leftarrow A, \Gamma_b, \Delta_b) \rightsquigarrow_{suc}^\varepsilon (\Gamma'_b, \Delta'_b)$ and

$$(C', \Gamma'_b, \Delta'_b) \rightsquigarrow_{adj} (\Gamma'_a, \Delta'_a)$$

of length m . If $V = \mathbf{u}$, then the induction step is completed by applying induction hypothesis for this derivation. Assume that $V = \mathbf{t}$. It is sufficient for the completion of the induction step to show that $A \in S_P(\Gamma_0^+ \cup \Delta_0^+)$. By Lemma 4.4, $A \in S_P(\Delta_b^+)$. Because $\Delta'_b \subseteq \Delta_0$ and S_P is monotonic, $A \in S_P(\Gamma_0^+ \cup \Delta_0^+)$.

(c) Assume that $(\leftarrow A, \Gamma_b, \Delta_b \cup \{A^*\}) \rightsquigarrow_{ff} (\Gamma'_b, \Delta'_b)$ and

$$(C' \cup \{(V, \leftarrow L_1 \dots L_{s-1} L_{s+1} \dots L_n)\}, \Gamma'_b, \Delta'_b) \rightsquigarrow_{adj} (\Gamma'_a, \Delta'_a)$$

of length m . By applying induction hypothesis for this derivation, the induction step is completed. q.e.d.