Algebraic and Topological Aspects of Rough Set Theory

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***Abstract*—The main purpose of this talk is to show how some widely known and well established algebraic and topological notions are closely related to notions and results introduced and rediscovered in the rough set literature.**

I. Introduction

Let $\mathcal{V}$ be a real vector space, that is, $\mathcal{V}$ is a nonempty set on which we have a structure consisting of two operations, addition of elements of $\mathcal{V}$ and multiplication of elements from $\mathcal{V}$ by real numbers. The properties of these two operations make it possible to define various classes of subsets of $\mathcal{V}$. For example, a set A in $\mathcal{V}$ is said to be convex when it has the following property: If $x \in A, y \in A$ and $0 < \alpha < 1$, then $(1-\alpha)x + \alpha y \in A$. Thus some sets in $\mathcal{V}$ are convex, some nonconvex, and nonconvex subsets can be approximated by convex ones for a number of purposes. Convex sets in real vector spaces are extremely useful, for example, in the theory of optimization and its various applications. However, no one has ever claimed that the theory of convex sets extends the classical theory of sets or is an alternative to it.

Consider now a much more simple structure $\langle \mathcal{U}, E \rangle$ where $\mathcal{U}$ is a nonempty set and E is an equivalence relation on $\mathcal{U}$. Using this simple structure, one can introduce, in different ways, the notion of a rough set in $\langle \mathcal{U}, E \rangle$. Some authors define rough sets as subsets of $\mathcal{U}$ with a certain property, some define them as certain pairs of subsets of $\mathcal{U}$, and some as certain collections of subsets of $\mathcal{U}$. Often, the terminology in young fields is not well established, and this seems to hold for the theory of rough sets as well. I am convinced, that in the standard set-theoretic framework, a rough set in $\langle \mathcal{U}, E \rangle$ should be defined not as a subset of $\mathcal{U}$ but as a subset of the power set of $\mathcal{U}$, and every subset of $\mathcal{U}$ should be a member of exactly one rough set.

In any case, when the notion of a rough set is introduced, the basic concepts from the ordinary set theory, like the relations of membership, equality, and subset, are employed in essential way. Nevertheless, surprisingly often, one can read in the literature that some subsets of $\mathcal{U}$ are rough, or that a rough set theory is an extension of classical set theory.

Equivalence relations appear naturally and have an important role in almost every field of mathematics. Therefore, we can expect that various notions and results of rough set theory have their counterparts in other well established and more developed areas of mathematics. Discovering and studying relationships between the rough set theory and another field cannot do any harm and often may be useful because such relationships can enrich both fields and may help to identify some underlying fundamental concepts and results.

The main purpose of this talk is to show how some widely known and well established algebraic and topological notions are closely related to notions and results introduced and rediscovered in the rough set literature.

II. Preliminaries

Throughout this paper we assume a modest familiarity with the standard concepts and basic facts of the ordinary set theory. Mostly we follow the terminology and notation of the book by P. R. Halmos [23].

If A is a set and a is an element of A, then we write $a \in A$; otherwise, we write $a \notin A$. Sets are considered *equal* if they have the same elements. If sets A and B are equal, we write $A = B$. If they are not equal, we write $A \neq B$. The empty set is denoted by $\emptyset$ and a set composed of one element only is called a *singleton*.

Sets can have other sets as elements. To avoid possible terminological monotony, we sometimes use the word *collection* synonymously with the word *set*. The collection of all subsets of a set A is denoted by $\mathcal{P}(A)$, and it is called the *power set* of A.

If A is a *subset* of B, that is, if every element of A is also an element of B, then we write $A \subset B$. If $A \subset B$ and if there is some element in B that does not belong to A, then A is called a *proper subset* of B[1]

The *union* of sets A and B, denoted by $A \cup B$, is the set of all elements that are either in A or in B. The unions of more than two sets are denoted in the following way. For example, $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$. Similarly, $\bigcup_{i=1}^{\infty} A_i$ and $\bigcup_{i \in I} A_i$.

[1]Warning: Some author use the symbols $A \subseteq B$ and $A \subset B$ to denote that "A is a subset of B" and "A is a proper subset of B," respectively.

The *intersection* of sets A and B, denoted by $A \cap B$, is the set of all elements that are in both A and B. If the intersection of sets A and B is the empty set, then A and B are said to be *disjoint*. For the intersections of more than two set, we use the notation $\bigcap_{i=1}^{n} A_i,\ \ \bigcap_{i=1}^{\infty} A_i,\ \ \bigcap_{i \in I} A_i$.

If A and B are sets, then the *relative complement* of B in A, denoted by $A \setminus B$, is the set of those elements in A that do not belong to B. The relative complement of B in A is also called the *difference* of sets A and B. A natural extension is the *symmetric difference*, which is the union of $A \setminus B$ with $B \setminus A$. We shall denote it by $A \Delta B$, that is, $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

If we are dealing with subsets of a particular set, we often call such a particular set the *universe of discourse* and simplify the notation for the relative complements of subsets with respect the universe of discourse. For example, if $\mathcal{U}$ is the universe of discourse and A is a subset of $\mathcal{U}$, then the difference $\mathcal{U} \setminus A$ is called the *complement* of A.

By a *partition* of a nonempty set we mean a collection of nonempty subsets that are disjoint from each other and whose union is the whole set. If P and Q are partitions of the same set, then the partition P is called a *refinement* of the partition Q if every member of P is a subset of one of the members of Q.

The order of elements in a set is important in many situations. If so, then we need an additional structure and notation. To indicate that we are dealing with ordering of elements of a finite set, we use parentheses instead of brackets and use the words *ordered tuples* or *finite sequences.* Two n-tuples $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ are said to be equal if $x_i = y_i$ for $1 \leq i \leq n$. The ordered 2-tuples are called ordered pairs.

If A and B are sets, then the *Cartesian product* of A and B, which is denoted by $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. The Cartesian product of sets $A_1, A_2, \dots, A_n$, in this order, is denoted by $A_1 \times A_2 \times \dots \times A_n$; that is, $A_1 \times A_2 \times \dots \times A_n = \{(x_1, x_2, \dots, x_n) : x_i \in A_i$ for $1 \leq i \leq n.\}$

By a *relation* between members of a set A and a set B, or a relation from A to B, we mean a subset of the Cartesian product $A \times B$. More generally, by an *n-ary relation* on (or over) the Cartesian product $A_1 \times A_2 \times \dots \times A_n$, we mean a subset of $A_1 \times A_2 \times \dots \times A_n$. If $A_1 = A_2 = \dots = A_n = A$, then we say *unary, binary, and ternary relation* on A instead of 1-ary, 2-ary, and 3-ary relation on A, $A \times A$, and $A \times A \times A$, respectively. If x and y are elements of some set X and $\mathcal{B}$ is a binary relation on X, then we also write $x \mathcal{B} y$ instead of $(x, y) \in \mathcal{B}$.

If $\mathcal{B}$ is a binary relation on a set X, then the *converse* $\mathcal{B}^{-1}$ of $\mathcal{B}$ is defined by $\mathcal{B}^{-1} = \{(x, y) : (y, x) \in \mathcal{B}\}$. The *identity* relation (also called the *diagonal*) on X is the set of all pairs of the form (x, x) for $x \in X$. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are binary relation on X, then the *composite* of $\mathcal{B}_1$ and $\mathcal{B}_2$ is denoted by $\mathcal{B}_1 \circ \mathcal{B}_2$; it is defined to be the set of all pairs (x, z) such that for some y it is true that $(x, y) \in \mathcal{B}_2$ and $(y, z) \in \mathcal{B}_1$. If $\mathcal{B}$ is a binary relation on a set X and if x is an element of X, then $\mathcal{B}(x)$ will denote the set $\{y \in X : (x, y) \in \mathcal{B})\}$.

A binary relation $\mathcal{B}$ on a set X is called *reflexive* if $x \mathcal{B} x$ for every $x \in X$. A reflexive binary relation on X is called *tolerance* if it is symmetric, a *preorder* if it transitive, an *equivalence* if it is transitive and symmetric.

Let us recall that partitions and equivalence relations are closely connected. Let $\mathcal{E}$ be the mapping from the set of all partitions of X into the set of binary relations on X defined as follows. If D is a partition of X, then $\mathcal{E}(D)$ is the binary relation such that $(x, y) \in \mathcal{E}(D)$ just in case x and y belong to the same member of partition D. It can easily be shown that $\mathcal{E}(D)$ is an equivalence relation on X.

On the other hand, every equivalence relation in X induces a partition of X as follows. If E is an equivalence relation on X, then an *equivalence class* with respect to E is defined as a subset A of X with the following two properties: (i) every two elements in A are equivalent, and (ii) each element of X that is equivalent to some element of A also belongs to A. It turns out that the collection of all equivalence classes with respect to E is a partition of X. By the *equivalence class of an element* x of X, we understand the equivalence class to which x belongs. The equivalence class of x will be denoted by x/E, and the collection of all equivalence classes of E will be denoted by $\mathcal{D}(E)$. Notice that $x/E = E(x)$.

For every equivalence E on X and every partition D of X, we have

$$\mathcal{D}(\mathcal{E}(D)) = D \quad \text{and} \quad \mathcal{E}(\mathcal{D}(E)) = E.$$

III. Approximation spaces

Many theoretical results in mathematics and many applications of mathematics in practice are based on the possibility of approximating subsets of a fixed set by other subsets. Sometimes approximations are needed because the complexity of objects under investigation necessitates simplification and various types of idealization, sometimes because insufficient information or other sources of uncertainty make it impossible to describe the sets in question precisely.

A frequent technique used for approximating subsets of some fixed set $\mathcal{U}$ by other subsets of $\mathcal{U}$ is to use mathematical structures available in $\mathcal{U}$ for finding a suitable pair f and g of mappings from the power set of $\mathcal{U}$ into itself such that, for every subset X of $\mathcal{U}$, the set $f(X)$ is included in X and X is included in $g(X)$. However, there are also situations in which this is too demanding or unnatural. Then, this condition can be weakened to requiring only that $f(X) \subset g(X)$ for every subset X of $\mathcal{U}$. Before proceeding we need some definitions.

Let $\mathcal{U}$ be a fixed nonempty set. For every set-to-set function $f : \mathcal{P}(\mathcal{U}) \to \mathcal{P}(\mathcal{U})$, we define the dual set-to-set function $f^d : \mathcal{P}(\mathcal{U}) \to \mathcal{P}(\mathcal{U})$ of f by

$$f^d(X) = \mathcal{U} \setminus f(\mathcal{U} \setminus X).$$

If we know the dual function f^d of f, then we can easily recover the original function f by

$$f(X) = \mathcal{U} \setminus f^d(\mathcal{U} \setminus X).$$

Thus it does not matter which of these functions is taken as primitive.

Because one of the most reasonable properties of any concept of approximation is that approximations of larger sets are larger or at least not smaller, we will be interested mainly in the set-to-set functions that are *isotonic* with respect to the partial order in $\mathcal{P}(\mathcal{U})$ given by the relation of set inclusion.

In this connection, we should notice that the following properties are mutually equivalent:

$$X \subset Y \text{ implies } f(X) \subset f(Y),$$

$$X \subset Y \text{ implies } f^d(X) \subset f^d(Y),$$

$$X \cap Y = \emptyset \text{ implies } f(X) \cap f^d(Y) = \emptyset,$$

$$f(X) \cup f(Y) \subset f(X \cup Y) \text{ for all } X, Y \in \mathcal{U},$$

$$f(X) \cap f(Y) \supset f(X \cap Y) \text{ for all } X, Y \in \mathcal{U},$$

$$f^d(X) \cup f^d(Y) \supset f^d(X \cup Y) \text{ for all } X, Y \in \mathcal{U},$$

$$f^d(X) \cap f^d(Y) \subset f^d(X \cap Y) \text{ for all } X, Y \in \mathcal{U}.$$

A thorough studies of isotonic functions can be found in a series of papers by Preston C. Hammer published in the beginning of the sixties of the last century; as examples, see [5] and [7]. For later investigation, see S. Gniłka [6] and B. M. R. Stadler & P. F. Stadler [9]. In what follows, we will use the terms introduced in Hammer's papers.

- The isotonic functions that are also non-shrinking in the sense that $X \subset f(X)$ for all X are called *expansive* functions.
- The isotonic functions that are also non-enlarging in the sense that $X \supset f(X)$ for all X are called *contractive* functions.
- The expansive functions that are also idempotent are called *closure* functions.
- The contractive functions that are also idempotent are called *interior* functions.

By an *approximation space* we understand in this paper a nonempty set $\mathcal{U}$ together with a pair (f, g) of functions from the power set of $\mathcal{U}$ into itself such that $f(X) \subset g(X)$ for every $X \subset \mathcal{U}$.

An approximation space $\langle \mathcal{U}, (f, g) \rangle$ is called

- *isotonic* if both f and g are isotone,
- *uniform* if $f(X) \subset X \subset g(X)$ for every $X \subset \mathcal{U}$.

A subset X of $\mathcal{U}$ will be called *exact* in approximation space $\langle \mathcal{U}, (f, g) \rangle$ if

$$f(X) = X = g(X).$$

IV. Pawlak's definable sets and rough sets

Consider information given by the following Table I, which is taken from Pawlak's example in [4]:

Patient	Headache	Muscular pain	Temperature	Flue
1	no	yes	high	yes
2	yes	no	high	yes
3	yes	yes	very high	yes
4	no	yes	normal	no
5	yes	no	high	no
6	no	yes	very high	yes

TABLE I

First, notice that some patients are indiscernible with respect to some attributes or sets of attributes. For instance, patients 2, 3 and 5 are indiscernible with respect to attribute "headache", and patients 3 and 6 are indiscernible with respect to the pair of attributes "muscular pain" and "flue".

Second, notice that every set of attributes divides the set of patients into a collection of pairwise disjoint nonempty subsets; that is, every set of attributes defines uniquely a partition of the set of patients. As examples, we have

Headache:	{1,4,6}, {2,3,5};
Temperature:	{1,2,5}, {3,6}, {4};
Muscular pain and Temperature:	{2,5}, {3,6}, {4}, {1}.

Let us consider some subset of patients, for example, the set $X = \{1, 3, 6\}$. It can be exactly described by means of the pair of attributes "muscular pain" and "temperature" as the set of the patients feeling muscular pain and having high temperature or very high temperature. However, it can easily be seen that X cannot be described exactly by using only the attribute "temperature".

A. Definable sets

Intuitively, these relations of indiscernibility lead us to considering as exactly describable or definable subsets (with respect to a given equivalence relation) only those that are equivalence classes or unions of some equivalence classes. Formally, we obtain the family of definable subsets considered by Pawlak as follows.

Let E be an equivalence relation in $\mathcal{U}$ and let $\mathcal{D}(E)$ be the partition of $\mathcal{U}$ induced by E. The lower and upper approximations of a subset X of $\mathcal{U}$ and definable subsets of $\mathcal{U}$ considered by Pawlak can be introduced as follows:

- The E-*lower approximation* of X is the union of those members of $\mathcal{D}(E)$ that are subsets of X.
- The E-*upper approximation* of X is the union of those members of $\mathcal{D}(E)$ that contain at least one element of X.
- A subset X of $\mathcal{U}$ is E-*definable* if it is either empty or a member of $\mathcal{D}(E)$ or the union of two or more members of $\mathcal{D}(E)$.

Put in other terms, the lower approximations of subsets of $\mathcal{U}$ are values of the set-to-set function $f_E : \mathcal{P}(\mathcal{U}) \to \mathcal{P}(\mathcal{U})$ defined by

$$f_E(X) = \bigcup \{A \in \mathcal{D}(E) : A \subset X\}.$$

It can easily be verified that, for the dual of f_E, we have

$$f_E^d(X) = \bigcup \{A \in \mathcal{D}(E) : A \cap X \neq \emptyset\}.$$

Thus the upper approximations of subsets of $\mathcal{U}$ are values of the function that is dual to the function whose values are lower approximations, and vice versa.

It can easily be seen that, for all subsets X and Y of $\mathcal{U}$, we have

- $f_E(X) \subset X \subset f_E^d(X)$,
- $X \subset Y$ implies $f_E(X) \subset f_E(Y)$ and $f_E^d(X) \subset f_E^d(Y)$,
- $f_E(f_E(X)) = f_E(X)$ and $f_E^d(f_E^d(X)) = f_E^d(X)$.
- $f_E(f_E(X)) = f_E^d(f_E(X))$.
- $f_E(f_E^d(X)) = f_E^d(f_E^d(X))$,

Now it is clear that

- $\langle \mathcal{U}, (f_E, f_E^d)\rangle$ is a uniform isotonic approximation space.
- f_E is an interior function; that is, f_E is idempotent, contractive, and isotonic.
- f_E^d is a closure function; that is, f_E^d idempotent, expansive, and isotonic.
- Pawlak's E-definable sets are exact in approximation space $\langle \mathcal{U}, (f_E, f_E^d)\rangle$.
- Pawlak's lower and upper approximations of subsets of $\mathcal{U}$ are E-definable sets.

It is also useful to notice that, for all X and Y,

- $f_E(X \cap Y) = f_E(X) \cap f_E(Y)$,
- $f_E^d(X \cup Y) = f_E^d(X) \cup f_E^d(Y)$,
- $f_E(\emptyset) = f_E^d(\emptyset) = \emptyset$ and $f_E(\mathcal{U}) = f_E^d(\mathcal{U}) = \mathcal{U}$.

B. Rough sets

There often exist different subsets of $\mathcal{U}$ with the property that their lower approximations are the same and at the same time their upper approximations are also the same. Such subsets are mutually indistinguishable in terms of approximations based on equivalence relations. It is therefore natural to use the lower and upper approximations defined by equivalence relation E for introducing a binary relation (let us denote it by $\equiv_E$) on the power set of $\mathcal{U}$ by requiring that $X \equiv_E Y$ holds for subsets X and Y of $\mathcal{U}$ if and only if

$$f_E(X) = f_E(Y) \text{ and } f_E^d(X) = f_E^d(Y).$$

It can easily be verified that this relation is an equivalence relation on the power set $\mathcal{P}(\mathcal{U})$. The equivalence classes of $\equiv_E$ are called the *rough sets* in $\langle \mathcal{U}, E\rangle$ or in $\langle \mathcal{U}, (f_E, f_E^d)\rangle$. To avoid possible misunderstanding it is useful to keep in mind that rough sets in $\langle \mathcal{U}, E\rangle$ are not subsets of $\mathcal{U}$ but certain sets of subsets of $\mathcal{U}$.

A particular rough set in $\langle \mathcal{U}, E\rangle$ can be represented by one of its members but it is not immediately clear whether there exists some natural or convenient rule for choosing such representation. However, J. Pomykała & J. A. Pomykała [11] have constructed a satisfactory representation; see also M. Gehrke & E. Walker [12].

Rough sets in $\langle \mathcal{U}, E\rangle$ can also be represented by the ordered pair $(f_E(X), f_E^d(X))$ where X is an arbitrary member of the rough set in question. This is possible because every equivalence class of $\equiv_E$ is uniquely determined by the pair $(f_E(X), f_E^d(X))$ where X can be any member of the class.

Before discussing relationships of rough set theory to other fields, we should mention that analogues of Pawlak's notions and constructions for equivalence relations have been introduced and studied for relations different from equivalences by a number of authors; for example, see Järvinen [14], Wybraniec-Skardovska [10], Järvinen & Kortelainen [13], Skowron & Stepaniuk [16], Pomykała [17], Düuntsch & Gediga [1], Yao [18] or Lin [19].

V. Links to other fields

Various equivalence relations and set approximations appear in almost every area of mathematics. It is therefore no surprise that some notions and results appearing in the literature on rough set theory have their counterparts in other fields. A noteworthy example of upper approximation from the classical mathematical analysis appears in the proof of Blaschke convergence theorem in Minkowski spaces, see Valentine [2]:

> Let A be a box[1], each edge of which has length τ, and which contains the collection $\mathcal{M}^2$ in its interior. Let P_i be the ith conventional gridlike partition of A into congruent boxes, each of which has edges of length $2^{-i}\tau$. The set of congruent boxes thus obtained from P_i is denoted by K_i. The union T of a subset of K_i will be called a *minimal covering* to a set $M \in \mathcal{M}$ if $M \subset T$ and if M intersects each box belonging to T.

Here the underlying relation is the equivalence E defined, loosely speaking, by the requirement that $(x, y) \in E$ if and only if x and y belong to the same box.

We will see that some parts of rough set theory and some parts of the theory of topological spaces are so closely related that the difference is only the question of translation between different languages. But first we will briefly discuss some algebraic aspects of rough set theory.

A. Links to algebra

The theory of rough sets is so close to some parts of algebra that some structures involving rough sets are called Pawlak's rough set algebras. The algebraic aspects of rough set theory have been studied by many authors in several past decades and there exist excellent papers dealing in great detail and deepness with the relationship between the theory of rough sets and some parts of algebra. The interested reader finds many of these result, for example, in [11], [20], [18], [21], [22] and the literature therein. Therefore, here we only wish to mention some reasons why this relationship is so intimate.

First we notice that the power set $\mathcal{P}(\mathcal{U})$ is partially ordered by the relation of set inclusion, and that this partially ordered set is a complete Boolean lattice.

Second, the set of all functions mapping the power set $\mathcal{P}(\mathcal{U})$ of nonempty set $\mathcal{U}$ into itself inherits from $\mathcal{P}(\mathcal{U})$ a set algebra and inclusion. Namely, $f_1 \cup f_2$, $f_1 \cap f_2$ and $f_1 \subset f_2$ are defined by the requirement that, for all $X \in \mathcal{P}(\mathcal{U})$,

$$(f_1 \cup f_2)(X) = f_1(X) \cup f_2(X),$$

$$(f_1 \cap f_2)(X) = f_1(X) \cap f_2(X),$$

$$f_1(X) \subset f_2(X).$$

[1] A box is an s-dimensional parallelepiped.
[2] $\mathcal{M}$ is a uniformly bounded infinite collection of closed convex sets.

Moreover, this set is also a semigroup with respect to the composition of functions and the identity function.

Third, the set of ordered pairs of functions from $\mathcal{P}(\mathcal{U})$ into itself can be partially ordered by the relation $\leq$ where $(f_1, f_2) \leq (g_1, g_2)$ means that

$$f_1(X) \subset g_1(X) \text{ and } f_2(X) \subset g_2(X)$$

for every subset X of $\mathcal{U}$.

We have already mentioned that Pawlak's rough sets can be represented by ordered pairs $(f_E(X), f_E^d(X))$ of lower and upper approximations. Therefore, the relation $\leq$ gives a natural partial order $\preceq$ on the collection of rough sets in $\langle \mathcal{U}, E \rangle$ by

$$(f_E(X), f_E^d(X)) \leq (f_E(Y), f_E^d(Y)).$$

As an illustration, consider the following collection of rough sets from a recent paper by Järvinen [20].

Let $\mathcal{U}$ be the set $\{a, b, c\}$ and let E be an equivalence on $\mathcal{U}$ whose equivalence classes are $\{a, c\}$ and $\{b\}$. Then the collection of rough sets consists of the following rough sets:

$$\{\emptyset\}, \{\{a\}, \{c\}\}, \{\{b\}\}, \{\{a, b\}, \{b, c\}\}, \{\{a, c\}\}, \{\mathcal{U}\}.$$

For example, we have

$$\{\{a\}, \{c\}\} \preceq \{\{a, c\}\} \text{ and } \{\{a\}, \{c\}\} \preceq \{\{a, b\}, \{b, c\}\}$$

because, for the corresponding pairs of lower and upper approximations, we have

$$(\emptyset, \{a, c\}) \leq (\{a, c\}, \{a, c\}) \text{ and } (\emptyset, \{a, c\}) \leq (\{b\}, \mathcal{U}).$$

It is known that this partially ordered set is a complete Stone lattice. It is also known (see, Järvinen [20]) that this is true also for the ordered set of rough sets determined analogously by relations that are simultaneously symmetric and transitive but not necessarily reflexive. However, Järvinen [14] has shown that the ordered set of rough sets determined by tolerance relations are not necessarily even semilattices, which is also true for transitive relations, see Järvinen [20].

B. Links to topology

The focus of the rest of the paper is the intimate connection between the approximation operators of rough set theory and operators commonly met in topological spaces and their generalizations. We have already mentioned that because of the monotonicity of Pawlak's lower and upper approximations with respect to set inclusion, one can find most of their properties in the context of Hammer's system of extended topology. The relationship between the theory of rough sets and theory of topological spaces was recognized by many authors already in the early days of rough set theory. Here we would like to attract attention also to the rarely noticed fact that topologies induced by approximation operators of rough set theory are the uniform topologies.

1) Topologies: A *topology* for $\mathcal{U}$ is a collection τ of subsets of $\mathcal{U}$ satisfying the following conditions:

- The empty set and $\mathcal{U}$ belong to τ.
- The union of the members of each sub-collection of τ is a member of τ.
- The intersection of the members of each finite sub-collection of τ is a member of τ.

If τ is a topology for $\mathcal{U}$ then we say that the pair $(\mathcal{U}, \tau)$ or simply $\mathcal{U}$ is a topological space. Let τ be a topology for $\mathcal{U}$ and X be a subset of $\mathcal{U}$.

- The members of τ are called the *open* sets of $(\mathcal{U}, \tau)$.
- The complements of the open sets are called the *closed* sets of $(\mathcal{U}, \tau)$.
- The *interior* of X is the largest open subset of X.
- The *closure* of X is the smallest closed subset that includes X.

2) Uniformities: A *quasiuniformity* for $\mathcal{U}$ is a nonempty collection ϱ of subsets of $\mathcal{U} \times \mathcal{U}$ such that

(a) Each member of ϱ contains the diagonal.
(b) The intersection of any two members of ϱ also belongs to ϱ.
(c) If R is a member of ϱ and S is a subset of $\mathcal{U} \times \mathcal{U}$ such that $R \subset S$, then S also belongs to ϱ.
(d) For each $R \in \varrho$ there is an $S \in \varrho$ such that $S \circ S \subset R$.

The quasiuniformity ϱ for $\mathcal{U}$ is called a *uniformity* for $\mathcal{U}$ if the following additional condition is satisfied:

(e) If R is in ϱ, then the inverse R^{-1} of R is also in ϱ.

If ϱ is a quasiuniformity or uniformity for $\mathcal{U}$, then the pair $(\mathcal{U}, \varrho)$ is said to be a *quasiuniform* or *uniform* space, respectively.

Every quasiuniformity ϱ on $\mathcal{U}$ yields a topology for $\mathcal{U}$ by taking as open sets the sets A with the property: if $x \in A$ then there is R in ϱ such that $\{y : (x, y) \in R\} \subset A$.

3) Topologies from equivalences: Let D be a partition of $\mathcal{U}$. It can easily be seen that the collection of all sets that can be written as unions of some members of D together with the empty set is a topology for $\mathcal{U}$. This topology is called the *partition topology* generated by D. The partition topologies are very special. They are characterized by the fact that every open set is also closed, and vice versa. Moreover, the partition topologies are Alexandrov topologies, which means that the intersection of the members of every (not only finite) collection of open sets is also open.

Because every equivalence E in $\mathcal{U}$ defines a partition of $\mathcal{U}$, it also generates a topology for $\mathcal{U}$; namely, the partition topology generated by the partition $\mathcal{D}(E)$. We denote it by τ_E and, if there is no danger of misunderstanding, we omit references to E.

In order to see clearly how Pawlak's approximation spaces $\langle \mathcal{U}, E \rangle$ are intimately related with the topological spaces, we observe that:

- A subset X of $\mathcal{U}$ is E-definable if and only if it is either empty or it can be written as the union of some members of the partition induced by E.

- A subset X of $\mathcal{U}$ is τ_E-open if and only if it is either empty or it can be written as the union of some members of the partition induced by E.

Moreover, if E is an equivalence on $\mathcal{U}$, then the collection ϱ of subsets of $\mathcal{U} \times \mathcal{U}$ defined by

$$\varrho = \{R : R \subset \mathcal{U} \times \mathcal{U}, E \subset R\}$$

is a uniformity for $\mathcal{U}$ and the topology for $\mathcal{U}$ induced by this uniformity coincides with topology τ_E.

Consequently,

- Pawlak's approximation spaces are uniform spaces whose uniform topologies coincide with partition topologies.
- These topologies can be characterized by the fact that the collection of open set coincides with the collection of closed sets.
- These topologies are the Alexandrov topologies.

Hence the difference between Pavlak's approximation space $\langle \mathcal{U}, E\rangle$ and the topological space (U, τ_E) is only terminological. In particular, we have

- X is definable if and only if it is open.
- X is definable if and only if it is closed.
- The lower approximation of X is the interior of X.
- The upper approximation of X is the closure of X.
- X is definable if and only if its interior is equal to its closure.

For translation of some other terms of Pawlak's terminology into the standard language of general topology, see [8].

Before proceeding to topologies derived from approximations based on more general relations, let us recall that we introduced Pawlak's upper approximation of X as the set $\bigcup\{A \in \mathcal{D}(E) : A \cap X \neq \emptyset\}$ where $\mathcal{D}(E)$ is the partition of $\mathcal{U}$ induced by equivalence E. The following equalities are true for every equivalence E:

$$\bigcup\{A \in \mathcal{D}(E) : A \cap X \neq \emptyset\} = \bigcup\{E(x) : x \in X\}$$

and

$$\bigcup\{A \in \mathcal{D}(E) : A \cap X \neq \emptyset\} = \{x : E(x) \cap X \neq \emptyset\}.$$

Moreover, we can replace E anywhere by its converse E^{-1} due to symmetry of equivalences. However, these properties are not necessarily valid for other types of relations on $\mathcal{U}$. As a consequence, when introducing analogues of Pawlak's approximations for other classes of relations, we have several choices that can lead to approximations with different properties.

4) Topologies from tolerances: If T is a transitive tolerance, then T is an equivalence and $T(x)$ is the equivalence class of x. As a consequence, for every x and y, if $T(x)$ and $T(y)$ are not equal, then they are disjoint. However, if a tolerance T is not transitive, then two distinct $T(x)$ and $T(y)$ may have common elements. Therefore, the collection $\{T(x) : x \in \mathcal{U}\}$ of subsets of $\mathcal{U}$ is not necessarily a partition of $\mathcal{U}$. By the reflexivity of T, $\{T(x) : x \in \mathcal{U}\}$ is always a covering of $\mathcal{U}$.

As an example, let us consider Table II taken from Järvinen [14]. From the table we can extract the tolerance relation $T = \Delta \cup A \cup B$ where Δ is the identity relation

Patient	Blood pressure	Hemoglobin	Temperature
1	103/65	125	39.3
2	97/60	116	39.1
3	109/71	132	39.2
4	150/96	139	37.1
5	145/93	130	37.3
6	143/95	121	37.8
7	138/83	130	36.7

TABLE II

in the set of patients, $A = \{(1,2),(2,1),(1,3),(3,1)\}$, and $B = (5,4),(4,5),(5,6),(6,5),(5,7),(7,5)\}$. The corresponding sets $T(x)$ are given in Table III.

$1/T$	$2/T$	$3/T$	$4/T$	$5/T$	$6/T$	$7/T$
{1,2,3}	{1,2}	{1,3}	{4,5}	{4,5,6,7}	{5,6}	{5,7}

TABLE III

Following Pawlak's approach, several authors have introduced the lower approximations, upper approximations of sets and definable sets with respect to tolerances. As examples, see Järvinen [14], Wybraniec-Skardovska [10], Järvinen & Kortelainen [13], Skowron & Stepaniuk [16], or Pomykała [17].

Let T be a tolerance relation on $\mathcal{U}$, and let the lower and upper approximations and definable sets be defined as follows.

- The *T-lower approximation* of X is the set of those elements x from $\mathcal{U}$ for which $T(x) \subset X\}$.
- The *T-upper approximation* of X is the set of those elements x from $\mathcal{U}$ for which $T(x) \cap X \neq \emptyset\}$.
- A subset X of $\mathcal{U}$ is *T-definable* if its T-lower and T-upper approximations coincide

Again it is true that, for the dual to the set-to-set functions f_T defined by

$$f_T(X) = \{x \in \mathcal{U} : T(x) \subset X\},$$

we have

$$f_T^d(X) = \{x \in \mathcal{U} : T(x) \cap X \neq \emptyset\}.$$

Because, for every equivalence E on $\mathcal{U}$,

$$\bigcup\{A \in \mathcal{D}(E) : A \cap X \neq \emptyset\} = \{x \in \mathcal{U} : E(x) \cap X \neq \emptyset\},$$

we immediately see that the functions f_T and f_T^d coincide with those introduced for equivalences, provided T is transitive.

It turns out that if T is a tolerance on $\mathcal{U}$, then

- $\langle \mathcal{U}, (f_T, f_T^d)\rangle$ is a uniform isotonic approximation space,
- the collection τ_T of T-definable sets is a topology for $\mathcal{U}$.
- the interior and closure of X relative to topology τ_T are equal to $f_T^d(f_T(X))$ and $f_T(f_T^d(X))$, respectively.

However, in general, the approximations defined by tolerances differ from those defined by equivalences. For example, the approximations $f_T(X)$ or $f_T^d(X)$ may be not T-definable, and f_T and f_T^d are not necessarily the interior or closure operators relative to topology τ_T.

Let E_T denote the intersection of all equivalence relations on $\mathcal{U}$ that include T. It turns out that E_T is an equivalence relation, and the collection of T-definable sets is the same as the collection of E_T-definable sets. Therefore, for each tolerance T, the collection of T-definable sets is a partition topology. Moreover, the collection ϱ of subsets of $\mathcal{U} \times \mathcal{U}$ defined by $\varrho = \{R : R \subset U \times U, T \subset R\}$ is a quasiuniformity for $\mathcal{U}$ and the topology for $\mathcal{U}$ induced by this quasiuniformity coincides with the partition topology generated by τ_{E_T}.

5) Topologies from preorders: Let Q be a preorder in $\mathcal{U}$ and x be an arbitrary point in $\mathcal{U}$. Let $Q^{\leftarrow}(x)$ and $Q^{\rightarrow}(x)$ be the subsets of $\mathcal{U}$ defined by

$$Q^{\leftarrow}(x) = \{y \in \mathcal{U} : (y, x) \in Q\},$$
$$Q^{\rightarrow}(x) = \{y \in \mathcal{U} : (x, y) \in Q\}.$$

If we wish to define the upper and lower and upper approximations with respect to preorders analogously to the cases of equivalences and tolerances, we have to take into account that preorders may be not symmetrical; that is, in general, $Q^{\leftarrow}(x) \neq Q^{\rightarrow}(x)$. Hence we have several possibly nonequivalent variants of definitions of the lower and upper approximations $\overline{Q}X$ and $\underline{Q}X$ of X. Namely,

$$\overline{Q}X = \{x \in \mathcal{U} : Q^{\rightarrow}(x) \cap X \neq \emptyset\} \quad (1)$$
$$\underline{Q}X = \{x \in \mathcal{U} : Q^{\rightarrow}(x) \subset X\} \quad (2)$$

$$\overline{Q}X = \{x \in \mathcal{U} : Q^{\leftarrow}(x) \cap X \neq \emptyset\} \quad (3)$$
$$\underline{Q}X = \{x \in \mathcal{U} : Q^{\leftarrow}(x) \subset X\} \quad (4)$$

$$\overline{Q}X = \{x \in \mathcal{U} : Q^{\rightarrow}(x) \cap X \neq \emptyset\} \quad (5)$$
$$\underline{Q}X = \{x \in \mathcal{U} : Q^{\leftarrow}(x) \subset X\} \quad (6)$$

$$\overline{Q}X = \{x \in \mathcal{U} : Q^{\leftarrow}(x) \cap X \neq \emptyset\} \quad (7)$$
$$\underline{Q}X = \{x \in \mathcal{U} : Q^{\rightarrow}(x) \subset X\} \quad (8)$$

We define the upper approximations $\overline{Q}X$ and lower approximations $\underline{Q}X$ of X by the following variant:

$$\overline{Q}X = \{x \in \mathcal{U} : Q^{\rightarrow}(x) \cap X \neq \emptyset\}$$
$$\underline{Q}X = \{x \in \mathcal{U} : Q^{\leftarrow}(x) \subset X\}$$

and we say that a subset X of $\mathcal{U}$ is *Q-definable* if

$$\underline{Q}X = \overline{Q}X.$$

To illustrate reasons for this choice, we consider the following example taken from Järvinen & Kortelainen [13].

Let $\mathcal{U}$ be the set $\{1, 2, 3\}$, and consider the preorder

$$Q = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 3)\}.$$

Then we obtain Table IV-Table VI. The last table shows that the richest collection of definable sets is obtained either for our choice or for the "dual" choice

x	$Q^{\leftarrow}(x)$	$Q^{\rightarrow}(x)$
1	{1}	{1,2,3}
2	{1,2}	{2,3}
3	{1,2,3}	{3}

TABLE IV
IMAGES AND INVERSE IMAGES UNDER Q

X	$\overline{Q}X, \rightarrow$	$\overline{Q}X, \leftarrow$	$\underline{Q}X, \rightarrow$	$\underline{Q}X, \leftarrow$
{1}	{1}	{1,2,3}	$\emptyset$	{1}
{2}	{1,2}	{2,3}	$\emptyset$	$\emptyset$
{3}	{1,2,3}	{3}	{3}	$\emptyset$
{1,2}	{1,2}	{1,2,3}	$\emptyset$	{1,2}
{1,3}	{1,2,3}	{1,2,3}	$\emptyset$	{1}
{2,3}	{1,2,3}	{2,3}	{2,3}	$\emptyset$

TABLE V
LOWER AND UPPER APPROXIMATIONS

$\underline{Q}\emptyset$	=	$\emptyset$	$\overline{Q}\emptyset$	=	$\emptyset$
$\underline{Q}\{1\}$	=	$\{1\}$	$\overline{Q}\{1\}$	=	$\{1\}$
$\underline{Q}\{2\}$	=	$\emptyset$	$\overline{Q}\{2\}$	=	$\{1, 2\}$
$\underline{Q}\{3\}$	=	$\emptyset$	$\overline{Q}\{3\}$	=	$\{1, 2, 3\}$
$\underline{Q}\{1, 2\}$	=	$\{1, 2\}$	$\overline{Q}\{1, 2\}$	=	$\{1, 2\}$
$\underline{Q}\{2, 3\}$	=	$\emptyset$	$\overline{Q}\{2, 3\}$	=	$\{1, 2, 3\}$
$\underline{Q}\{1, 3\}$	=	$\{1\}$	$\overline{Q}\{1, 3\}$	=	$\{1, 2, 3\}$
$\underline{Q}\{1, 2, 3\}$	=	$\{1, 2, 3\}$	$\overline{Q}\{1, 2, 3\}$	=	$\{1, 2, 3\}$

TABLE VI
LOWER AND UPPER APPROXIMATIONS FOR (5) AND (6)

$$\overline{Q}X = \{x \in U : Q^{\leftarrow}(x) \cap X \neq \emptyset\}$$
$$\underline{Q}X = \{x \in U : Q^{\rightarrow}(x) \subset X\}.$$

Let Q be a preorder in $\mathcal{U}$. Järvinen & Kortelainen [13] showed that the collection of Q-definable sets is again a topology for $\mathcal{U}$. We denote this topology by τ_Q. This topology is not necessarily a partition topology but it is always an Alexandrov topology.

Conversely, let τ be an Alexandrov topology for $\mathcal{U}$, and let Q_τ be a binary relation on $\mathcal{U}$ defined by $(x, y) \in Q_\tau$ if and only if $N_x \subset N_y$ where N_x and N_y are the neighborhood systems of x and y in τ, respectively.

It turns out that τ_Q is an Alexadrov topology if Q is a preorder, Q_τ is a preorder if τ is an Alexandrov topology, and

$$\tau_{Q_\tau} = \tau \text{ and } Q_{\tau_Q} = Q.$$

It is worth mentioning the situation for the case of arbitrary relations. Let R be an arbitrary binary relation on $\mathcal{U}$ and let $f_R : \mathcal{P}(\mathcal{U}) \to \mathcal{P}(\mathcal{U})$ be the set-to-set function defined by

$$f_R(X) = \bigcup\{R^{-1}(x) : R^{-1}(x) \subset X\}.$$

The dual of f_R is then given by

$$f_R^d(X) = \{x \in \mathcal{U} : R(x) \subset \bigcup\{R(y) : y \in X\}.$$

It can be showed that, for an arbitrary relation R on $\mathcal{U}$, $\langle \mathcal{U}, (f_R, f_R^d)\rangle$ is an isotonic approximation space, f_R is an interior function and f_R^d is a closure function in terms of Hammer's system of extended topology. As pointed out by Düntch and Gediga [1], if R is reflexive, then

$$f(X) \subset f_R(X) \subset X \subset f_R^d(X) \subset g(X)$$

for most of other approximations pairs (f, g) considered in the literature. In this sense, the pair (f_R, f_R^d) provides the tightest approximations.

Given that equivalences and tolerances are symmetric, it is immediate that these approximations, and consequently the collections of definable sets, coincide with those described above for equivalences, tolerances and preorders.

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