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Random walks and isotropic Markov chains on homogeneous spaces

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Abstract

Let G be a topological group acting on S transitively from the left with a compact stabilizer K. We show that every isotropic (i.e. spatially homogeneous w.r.t. the G-actions) Markov chain on S can be lifted to a right random walk on G and give a one-to-one correspondence between the isotropic Markov chains on S and the totality of sequences of probabilities $(\nu, \mu_1, \mu_2, \cdots)$ where ν is a probability on G/K and each μ_n is that on $K\backslash G/K$.

KEYWORDS: random walk, Markov chain

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1. Introduction and Main Result

In order to describe our problem, let us recall a well-known property of isotropic Markov chains on $\boldsymbol{R}^d$. We mean by a Markov chain a Markov process with discrete time parameter ($\in \boldsymbol{N} = \{0, 1, 2, \cdots\}$) and with an arbitrary state space. We identify those Markov chains which are equivalent in law. Let $M = (M_n)_{n\in\boldsymbol{N}}$ be a Markov chain on $\boldsymbol{R}^d$ with transition probability $P_n^M(x, E)$ ($x \in \boldsymbol{R}^d$, $E \in \mathcal{B}(\boldsymbol{R}^d)$, $n \in \boldsymbol{N}_+$) and with initial law $\mathbf{P}^{M_0}$, i.e. $\mathbf{P}(M_n \in E | M_{n-1} = x) = P_n^M(x, E)$ and $\mathbf{P}(M_0 \in E) = \mathbf{P}^{M_0}(E)$. Here $\mathbf{P}$ is a probability and $\mathcal{B}(\boldsymbol{R}^d)$ denotes the Borel subsets of $\boldsymbol{R}^d$ and $\boldsymbol{N}_+ = \boldsymbol{N} \setminus \{0\}$. M is called isotropic if

$$P_n^M(x, E) = P_n^M(x + y, E + y) \qquad \text{for } n \in \boldsymbol{N}_+, x, y \in \boldsymbol{R}^d, E \in \mathcal{B}(\boldsymbol{R}^d) \tag{1}$$

holds. Then, it is well-known that M_n is expressed as a sum of independent random variables, namely there exist $\boldsymbol{R}^d$-valued independent random variables $Y_0, Y_1, Y_2, \cdots$ such that $M_n = Y_0 + Y_1 + \cdots + Y_n$. M is thus regarded as a random walk on $\boldsymbol{R}^d$ (in a little wider sense). Put $\mu_n = \mathbf{P}^{Y_n}$ ($n \in \boldsymbol{N}_+$) and $\nu = \mathbf{P}^{Y_0}$. Then $P_n^M(x, E) = \mathbf{P}(Y_0 + \cdots + Y_n \in E | Y_0 + \cdots + Y_{n-1} = x) = \mu_n(E - x)$. Hence we have one-to-one correspondence between

$$\{\textit{isotropic Markov chain on } \boldsymbol{R}^d\} \qquad \text{and} \qquad \{(\nu, \mu_1, \mu_2, \cdots); \nu, \mu_n \textit{ are probabilities on } \boldsymbol{R}^d\}\ .$$

$\boldsymbol{R}^d$ acts on $\boldsymbol{R}^d$ transitively as translations: $x \mapsto x + y$. Isotropy 1 of a Markov chain is the invariance of the transition probabilities with respect to these actions. In this note we show that a similar one-to-one correspondence as above holds also in a noncommutative situation.

Let G be a topological group acting on a set S transitively from the *left*. We consider a Markov chain $M = (M_n)_{n\in\boldsymbol{N}}$ on S with transition probability $P_n^M(x, A)$ ($x \in S, A \in \mathcal{B}(S), n \in \boldsymbol{N}_+$).

Definition M is called isotropic if its transition probabilities satisfy

$$P_n^M(gx, gA) = P_n^M(x, A) \qquad \text{for} \quad n \in \boldsymbol{N}_+,\ x \in S,\ g \in G,\ A \in \mathcal{B}(S)\ . \tag{2}$$

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Let us fix a point $x_0 \in S$ and put $K = \{g \in G; gx_0 = x_0\}$ (the stabilizer of x_0). Throughout this note we assume that K is *compact* in G. Denote by π the canonical projection of G onto G/K ($\simeq S$) and by σ that of G/K onto $G//K$ where we write $G//K$ instead of $K\backslash G/K$ to denote a double coset space.

Theorem 1 *A one-to-one correspondence between*

$$\begin{aligned}\mathcal{M} &= \{\text{isotropic Markov chain on } S\} \qquad \text{and}\\ \mathcal{P} &= \{(\nu, \mu_1, \mu_2, \cdots);\ \nu \text{ is a probability on } G/K,\ \mu_n \text{ is a probability on } G//K\}\end{aligned}$$

exists. The correspondence is given as follows.
[$\mathcal{P} \longrightarrow \mathcal{M}$] For given $(\nu, \mu_1, \mu_2, \cdots) \in \mathcal{P}$, lifting canonically (and uniquely) ν and μ_n to K-right invariant probability $\tilde{\nu}$ and K-bi-invariant one $\tilde{\mu}_n$ on G respectively and taking G-valued independent random variables $Y_0, Y_1, Y_2, \cdots$ such that $\tilde{\nu} = \mathbf{P}^{Y_0}$ and $\tilde{\mu}_n = \mathbf{P}^{Y_n}$, we have $(\pi(Y_0Y_1\cdots Y_n))_{n\in\mathbf{N}} \in \mathcal{M}$.
[$\mathcal{M} \longrightarrow \mathcal{P}$] For given $M = (M_n) \in \mathcal{M}$, we have

$$\mu_n(\Lambda) = P_n^M(x_0, \sigma^{-1}\Lambda) \quad \text{for } \Lambda \in \mathcal{B}(G//K) \qquad \text{and} \qquad \nu = \mathbf{P}^{M_0}\ . \tag{3}$$

Before proving Theorem 1 (in the next section), we note some remarks on Theorem 1.

Remark 1 Taking the left and right random walks on G consisting of $Y_0, Y_1, Y_2, \cdots$ in Theorem 1, we get two systems of random variables on G/K: $L_n = \pi(Y_n \cdots Y_1 Y_0)$, $M_n = \pi(Y_0 Y_1 \cdots Y_n)$. Since $L_n = Y_n \cdots Y_1 \pi(Y_0)$, (L_n) is clearly a Markov chain on G/K. As the following simple example shows, however, (L_n) is not isotropic with respect to the left actions of G (even if (G, K) is a Gel'fand pair), while (M_n) is an isotropic Markov chain on G/K (Lemma 1, 3, 4).

Example Let $G = \mathcal{S}_3$, $S = \{1,2,3\}$ and $x_0 = 1$. Then, $K = \{1, (2\,3)\}$, $G//K = \{R_0, R_1\}$ where $R_0 = K$, $R_1 = \{(1,2), (1\,3), (1\,2\,3), (1\,3\,2)\}$. Put $\mu(R_0) = a$, $\mu(R_1) = b$. Then its K-bi-invariant lift $\tilde{\mu}$ satisfies $\tilde{\mu}(1) = \tilde{\mu}((2\,3)) = a/2$, $\tilde{\mu}((1\,2)) = \tilde{\mu}((1\,3)) = \tilde{\mu}((1\,2\,3)) = \tilde{\mu}((1\,3\,2)) = b/4$. Put $\nu = \delta_1$ on S and hence $\tilde{\nu}(1) = \tilde{\nu}((2\,3)) = 1/2$. Take $L_n = \pi(Y_n \cdots Y_1 Y_0)$ as in above Remark, and its transition probability $P_n^L(x, y)$ satisfies:

$$\begin{aligned}P_n^L(1,2) &= \tilde{\mu}(\{(1\,2), (1\,2\,3)\}) = b/2\ ,\\ P_n^L((1\,2\,3)1, (1\,2\,3)2) &= P_n^L(2,3) = \tilde{\mu}(\{(2\,3), (1\,2\,3)\}) = a/2\ +\ b/4\ .\end{aligned}$$

Hence (L_n) is not isotropic unless $2a = b$.

Remark 2 Furstenberg defined a discrete Brownian motion on a homogeneous space G/K in [2] as follows. Let μ be a K-left invariant absolutely continuous probability on G, $Y_1, Y_2, \cdots$ G-valued i.i.d. random variables with law μ and Y_0 another G-valued random variable independent of Y_i's. Put $W_n = Y_0 Y_1 \cdots Y_n$ (right random walk) and $U_n = \pi(W_n)$ where $\pi : G \longrightarrow G/K$. $(U_n)_{n\in\mathbf{N}}$ is called a discrete Brownian motion on G/K. Note that K-left invariance of μ is sufficient to ensure the Markovian property of (U_n) (See Lemma 1, 3). Theorem 1 immediately yields the following characterization.

Corollary 1 *The discrete Brownian motions on G/K coincide with the temporally homogeneous isotropic Markov chains on G/K with absolutely continuous transition probabilities.*
This fact is already shown by Chassaing (See [1]).

Remark 3 Denote by $M^b(G//K)$ the bounded measure algera on $G//K$ with convolution products. For $\mu \in M^b(G//K)$ and $\lambda \in M^b(G/K)$ (the linear space of bounded measures on G/K), we have $\lambda * \mu \in M^b(G/K)$. Let $M = (M_n)$ be an isotropic Markov chain on $S = G/K$ and $(\nu, \mu_1, \mu_2, \cdots)$ its corresponding sequence of probabilities described in Theorem 1. Then n-fold transition probability $P^{(n)}(x_0, A)$ is given by

$$\int_S \cdots \int_S P_1^M(x_0, dx_1) \cdots P_{n-1}^M(x_{n-2}, dx_{n-1}) P_n^M(x_{n-1}, A) = \delta_{x_0} * \mu_1 * \cdots * \mu_n\ . \tag{4}$$

If (G, K) is a Gel'fand pair therefore $M^b(G//K)$ is commutative, we thus see that Y_n's in Theorem 1 satisfy

$$M_n = \pi(Y_0Y_1\cdots Y_n) = \pi(Y_0Y_{\tau(1)}\cdots Y_{\tau(n)}) \quad \text{(in law)} \qquad \text{for } n \in \boldsymbol{N}_+, \tau \in \mathcal{S}_n$$

though $Y_0Y_1\cdots Y_n = Y_0Y_{\tau(1)}\cdots Y_{\tau(n)}$ (in law) may not hold. For measure algebras on Gel'fand pairs, see e.g. [3] and [4].

2. Proof of Theorem 1

Throughout this section we use the following notations.

- $X = (X_n)_{n\in\boldsymbol{N}}$: a Markov chain on G
- $M = (M_n)_{n\in\boldsymbol{N}}$: a Markov chain on G/K
- $P_n^X(g, E) = \mathbf{P}(X_n \in E|X_{n-1} = g)$: the n-th transition probability of X on G
- $P_n^M(x, A) = \mathbf{P}(M_n \in A|M_{n-1} = x)$: the n-th transition probability of M on G/K
- $\mathbf{P}^{X_0} = \mathbf{P}(X_0 \in \cdot\,)$: the law of X_0
- $(Y_n)_{n\in\boldsymbol{N}}$: G-valued independent random variables
- μ_n : a probability on $G//K$
- ν : a probability on G/K
- $\tilde{\mu}_n, \tilde{\nu}$: probabilities on G
- $\pi : G \longrightarrow G/K$, $\sigma : G/K \longrightarrow G//K$: the canonical projections
- $\pi_*\tilde{\mu}$: the push-forward of $\tilde{\mu}$ by π onto G/K
- l_x : the uniform probability supported by fibre $\pi^{-1}x(\subset G)$ for $x \in G/K$

If $P_n^{\pi X} = P_n^M$ $(\forall n \in \boldsymbol{N}_+)$ and $\mathbf{P}^{\pi X_0} = \mathbf{P}^{M_0}$ hold, we call $X = (X_n)$ a lift of $M = (M_n)$. We prove Lemma 1–6 and then complete the proof of Theorem 1.

Lemma 1 *If Markov chain $X = (X_n)$ on G satisfies*

$$P_n^X(g, \pi^{-1}A) = P_n^X(gk, \pi^{-1}A) \quad \text{for} \quad n \in \boldsymbol{N}_+, g \in G, k \in K, A \in \mathcal{B}(G/K), \tag{5}$$

then $\pi X = (\pi X_n)$ is also a Markov chain on G/K and satisfies

$$P_n^{\pi X}(\pi g, A) = P_n^X(g, \pi^{-1}A) \quad \text{for} \quad n \in \boldsymbol{N}_+, g \in G, A \in \mathcal{B}(G/K)\ . \tag{6}$$

Proof We denote by $\mathcal{E}_n, \mathcal{F}_n, \mathcal{E}^{(n)}$ and $\mathcal{F}^{(n)}$ the σ-algebras generated by $\{X_0, \cdots, X_n\}$, $\{\pi X_0, \cdots, \pi X_n\}$, $\{X_n\}$ and $\{\pi X_n\}$ respectively. Condition 5 is equivalent to $\mathcal{F}^{(n)}$-measurability of $\mathbf{E}[\phi(\pi X_{n+1})|\mathcal{E}^{(n)}]$ for any bounded measurable function ϕ on G/K. Hence, using Markovian property of (X_n), we have

$$\begin{aligned}\mathbf{E}[\phi(\pi X_{n+1})|\mathcal{F}_n] &= \mathbf{E}[\mathbf{E}[\phi(\pi X_{n+1})|\mathcal{E}_n]|\mathcal{F}_n] = \mathbf{E}[\mathbf{E}[\phi(\pi X_{n+1})|\mathcal{E}^{(n)}]|\mathcal{F}_n] \\ &= \mathbf{E}[\mathbf{E}[\phi(\pi X_{n+1})|\mathcal{F}^{(n)}]|\mathcal{F}_n] = \mathbf{E}[\phi(\pi X_{n+1})|\mathcal{F}^{(n)}]\ .\end{aligned}$$

$\mathcal{F}^{(n)}$-measurability of $\mathbf{E}[1_A(\pi X_{n+1})|\mathcal{E}^{(n)}]$ implies

$$\mathbf{E}[1_A(\pi X_{n+1})|\pi X_n = \pi g] = \mathbf{E}[1_{\pi^{-1}A}(X_{n+1})|X_n = g]\ ,$$

i.e. 6.

QED

Lemma 2 *Every Markov chain (M_n) on G/K admits a unique lift (X_n) on G satisfying*

$$P_n^X(g, E) = P_n^X(g, Ek) \quad \text{for} \quad g \in G, k \in K, E \in \mathcal{B}(G) \tag{7}$$
$$\mathbf{P}^{X_0}(E) = \mathbf{P}^{X_0}(Ek) \quad \text{for} \quad k \in K, E \in \mathcal{B}(G) \tag{8}$$

(i.e. K-right invariance of the transition probabilities and the initial law) and 5.

Proof Put

$$P_n^X(g, E) = \int_{G/K} l_x(E) P_n^M(\pi g, dx) \quad \text{for} \quad g \in G, E \in \mathcal{B}(G) , \tag{9}$$
$$\mathbf{P}^{X_0}(E) = \int_{G/K} l_x(E) \mathbf{P}^{M_0}(dx) \quad \text{for} \quad E \in \mathcal{B}(G) . \tag{10}$$

Markov chain $X = (X_n)$ defined by 9 and 10 obviously satisfies 5, 7 and 8. (X_n) is a lift of (M_n) since

$$\mathbf{P}^{\pi X_0}(A) = \int_{G/K} l_y(\pi^{-1}A)\mathbf{P}^{M_0}(dy) = \int_{G/K} 1_A(y)\mathbf{P}^{M_0}(dy) = \mathbf{P}^{M_0}(A) ,$$
$$P_n^{\pi X}(x, A) = P_n^X(g, \pi^{-1}A) = \int_{G/K} l_y(\pi^{-1}A) P_n^M(x, dy) = P_n^M(x, A)$$

where $x = \pi g \in G/K$, $A \in \mathcal{B}(G/K)$.

Combining K-right invariance 7 and 8 with $P_n^{\pi X} = P_n^M$ and $\mathbf{P}^{\pi X_0} = \mathbf{P}^{M_0}$, we see that P_n^X and $\mathbf{P}^{X_0}$ must have the expression 9 and 10 respectively (by virtue of the uniqueness of disintegration induced by $\pi : G \longrightarrow G/K$). This implies the uniqueness of lifting.

QED

Let $W_n = Y_0 Y_1 \cdots Y_n$ be a right random walk on G with $\mathbf{P}^{Y_0} = \tilde{\nu}$ and $\mathbf{P}^{Y_n} = \tilde{\mu}_n$. We call $W = (W_n)_{n \in \boldsymbol{N}}$ the right random walk on G generated by $(\tilde{\nu}, \tilde{\mu}_n, n \in \boldsymbol{N}_+)$.

Lemma 3 *If every $\pi_* \tilde{\mu}_n$ is K-invariant (in particular if every $\tilde{\mu}_n$ is K-left invariant), then P_n^W satisfies 5.*

Proof Since (W_n) is generated by $(\tilde{\nu}, \tilde{\mu}_n, n \in \boldsymbol{N}_+)$, its transition probability is given by

$$P_n^W(g, E) = \tilde{\mu}_n(g^{-1}E) \quad \text{for} \quad n \in \boldsymbol{N}_+, g \in G, E \in \mathcal{B}(G) . \tag{11}$$

In particular,

$$P_n^W(g, \pi^{-1}A) = \tilde{\mu}_n(g^{-1}\pi^{-1}A) = \pi_*\tilde{\mu}_n(g^{-1}A) = \pi_*\tilde{\mu}_n(k^{-1}g^{-1}A) = P_n^W(gk, \pi^{-1}A)$$

for $A \in \mathcal{B}(G/K)$ and $k \in K$.

QED

From Lemma 1 and Lemma 3, a right random walk (W_n) on G generated by $(\tilde{\nu}, \tilde{\mu}_n, n \in \boldsymbol{N}_+)$ such that $\pi_*\tilde{\mu}_i$ is K-invariant can be projected to a Markov chain (πW_n) on G/K with transition probability

$$P_n^{\pi W}(\pi g, A) = \pi_*\tilde{\mu}_n(g^{-1}A) \quad \text{for} \quad n \in \boldsymbol{N}_+, g \in G, A \in \mathcal{B}(G/K), \tag{12}$$

which satisfies:

Lemma 4 *(πW_n) is an isotropic Markov chain i.e. satisfies 2.*

Proof We get from 12

$$P_n^{\pi W}(g\pi h, gA) = P_n^{\pi W}(\pi(gh), gA) = \pi_*\tilde{\mu}_n((gh)^{-1}gA) = P_n^{\pi W}(\pi h, A)$$

for $g, h \in G$ and $A \in \mathcal{B}(G/K)$.

QED

Lemma 5 *For probability $\tilde{\mu}$ on G, K-right invariance of $\tilde{\mu}$ and K-invariance of $\pi_*\tilde{\mu}$ imply K-bi-invariance of $\tilde{\mu}$.*

Proof Since $\tilde{\mu}$ is K-right invariant, it is expressed as $\tilde{\mu} = \int_{G/K} l_x \pi_* \tilde{\mu}(dx)$. Denote by $k_*\tilde{\mu}$ the push-forward of $\tilde{\mu}$ by the map $x \in G \mapsto kx \in G$. Then

$$k_*\tilde{\mu} = \int_{G/K} k_* l_x \pi_* \tilde{\mu}(dx) = \int_{G/K} l_{kx} \pi_* \tilde{\mu}(dx) = \int_{G/K} l_y \pi_* \tilde{\mu}(dy) = \tilde{\mu} ,$$

(The third equality follows from K-invariance of $\pi_*\tilde{\mu}$.) i.e. $\tilde{\mu}$ is K-left invariant.

QED

Lemma 6 *Let $X = (X_n)_{n\in \boldsymbol{N}}$ be an isotropic Markov chain on G. Put $\tilde{\nu} = \mathbf{P}^{X_0}$, $\tilde{\mu}_n = P_n^X(e, \cdot)$ and $Y_0 = X_0, Y_1 = X_0^{-1}X_1, \cdots, Y_n = X_{n-1}^{-1}X_n, \cdots$. Then $\{Y_n\}_{n\in \boldsymbol{N}}$ is a system of G-valued independent random variables satisfying $\mathbf{P}^{Y_n} = \tilde{\mu}_n$ for $\forall n \in \boldsymbol{N}_+$. Thus X is a right random walk on G generated by $(\tilde{\nu}, \tilde{\mu}_n, n \in \boldsymbol{N}_+)$.*

Proof For $E \in \mathcal{B}(G)$,

$$\begin{aligned}
\mathbf{P}^{Y_n}(E) &= \mathbf{P}(Y_n \in E) = \int_G \mathbf{P}(Y_n \in E | X_{n-1} = g)\mathbf{P}^{X_{n-1}}(dg) \\
&= \int_G \mathbf{P}(X_n \in gE | X_{n-1} = g)\mathbf{P}^{X_{n-1}}(dg) = \int_G P_n^X(g, gE)\mathbf{P}^{X_{n-1}}(dg) \\
&= \int_G P_n^X(e, E)\mathbf{P}^{X_{n-1}}(dg) = P_n^X(e, E) = \tilde{\mu}_n(E) . \qquad (13)
\end{aligned}$$

We show that $\{Y_0, Y_1, \cdots, Y_n\}$ is an independent system for $\forall n \in \boldsymbol{N}$ using induction on n. Assuming $\mathbf{P}^{(Y_0, Y_1, \cdots, Y_{n-1})} = \tilde{\nu} \times \tilde{\mu}_1 \times \cdots \times \tilde{\mu}_{n-1}$, we have, for $E_0, E_1, \cdots, E_n \in \mathcal{B}(G)$,

$$\begin{aligned}
&\mathbf{P}(Y_0 \in E_0, \cdots, Y_n \in E_n) \\
=& \int_{E_0 \times \prod_{i=1}^{n-1} E_i} \mathbf{P}(Y_n \in E_n | Y_0 = g_0, \cdots, Y_{n-1} = g_{n-1})\mathbf{P}^{(Y_0, \cdots, Y_{n-1})}(dg_0 \cdots dg_{n-1}) \\
=& \int_{E_0 \times \prod_{i=1}^{n-1} E_i} \mathbf{P}(X_n \in g_0 \cdots g_{n-1}E_n | X_{n-1} = g_0 \cdots g_{n-1})\tilde{\nu} \times \prod_{i=1}^{n-1} \tilde{\mu}_i(dg_0 dg_1 \cdots dg_{n-1}) \\
=& \int_{E_0 \times \prod_{i=1}^{n-1} E_i} P_n^X(e, E_n)\tilde{\nu} \times \prod_{i=1}^{n-1} \tilde{\mu}_i(dg_0 dg_1 \cdots dg_{n-1}) \quad \text{(by isotropy 2)} \\
=& \tilde{\nu}(E_0) \prod_{i=1}^{n} \tilde{\mu}_i(E_i) .
\end{aligned}$$

Hence $\mathbf{P}^{(Y_0, Y_1, \cdots, Y_n)} = \tilde{\nu} \times \prod_{i=1}^{n} \tilde{\mu}_i$.

QED

Proof of Theorem 1 Denote the correspondence described in Theorem 1 by $\Phi : \mathcal{P} \longrightarrow \mathcal{M}$ and $\Psi : \mathcal{M} \longrightarrow \mathcal{P}$. We show $\Psi \circ \Phi = id_{\mathcal{P}}$ and $\Phi \circ \Psi = id_{\mathcal{M}}$. (Recall that Markov chains equivalent in law are identified.)

For $(\nu, \mu_n, n \in \boldsymbol{N}_+) \in \mathcal{P}$, we take G-valued independent random variables $Y_0, Y_1, Y_2, \cdots$ defined in Theorem 1. Put $X_n = Y_0 Y_1 \cdots Y_n$. We see $(\pi X_n)_{n\in \boldsymbol{N}} \in \mathcal{M}$ from Lemma 3, Lemma 1 and Lemma 4. Moreover, 12 yields for $\Lambda \in \mathcal{B}(G//K)$

$$P_n^{\pi X}(x_0, \sigma^{-1}\Lambda) = P_n^{\pi X}(\pi e, \sigma^{-1}\Lambda) = \pi_*\tilde{\mu}_n(\sigma^{-1}\Lambda) = \mu_n(\Lambda) .$$

(The last equality follows from K-bi-invariance of $\tilde{\mu}_n$.) We thus obtain $\Psi \circ \Phi = id_{\mathcal{P}}$.

For $M = (M_n)_{n\in \boldsymbol{N}} \in \mathcal{M}$, we take its lift $X = (X_n)_{n\in \boldsymbol{N}}$ described in Lemma 2 (explicitly by 9 and 10). X is isotropic since we have, for $g, g' \in G$ and $E \in \mathcal{B}(G)$,

$$\begin{aligned} P_n^X(g'g, g'E) &= \int_{G/K} l_x(g'E) P_n^M(\pi(g'g), dx) = \int_{G/K} l_{g'^{-1}x}(E) P_n^M(g'\pi g, dx) \\ &= \int_{G/K} l_y(E) P_n^M(g'\pi g, g'dy) = \int_{G/K} l_y(E) P_n^M(\pi g, dy) = P_n^X(g, E) \end{aligned}$$

using 9 and isotropy 2 of M. Hence X is a right random walk on G by Lemma 6. Let $(\tilde{\nu}, \tilde{\mu}_n, n \in \boldsymbol{N}_+)$ be the sequence which generates X (defined in Lemma 6). In order to see $\Phi \circ \Psi(M) = M$, it suffices to show that $\tilde{\mu}_n$ [resp. $\tilde{\nu}$] is the K-bi-invariant [resp. K-right invariant] lift of μ_n [resp. ν] defined by 3. The assertion for $\tilde{\nu}$ is trivial (by 10). $\tilde{\mu}_n = P_n^X(e, \cdot)$ is K-right invariant by 9. Moreover, 12 yields for $A \in \mathcal{B}(G/K)$ and $k \in K$

$$\pi_* \tilde{\mu}_n(A) = P_n^M(x_0, A) = P_n^M(kx_0, kA) = P_n^M(x_0, kA) = \pi_* \tilde{\mu}_n(kA) \ .$$

Hence $\tilde{\mu}_n$ is K-bi-invariant by Lemma 5. Using 12 again, we obtain

$$(\sigma\pi)_* \tilde{\mu}_n(\Lambda) = \pi_* \tilde{\mu}_n(\sigma^{-1}\Lambda) = P_n^M(x_0, \sigma^{-1}\Lambda) = \mu_n(\Lambda)$$

for $\Lambda \in \mathcal{B}(G//K)$. This completes the proof of $\Phi \circ \Psi = id_{\mathcal{M}}$.

QED

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