

Modeling of fiber reinforced concrete by the homogenization method

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The simulation of mechanical behavior of steel fiber reinforced concrete SFRC is introduced utilizing the homogenization method. The random distribution of fibers has been converted to a periodic distribution. Using the periodicity assumption, the boundary conditions for the unit cell are derived. The homogenized stiffness matrix is determined in elastic and plastic range. A numerical example to study the effect of the volume fraction of fibers is introduced.

Key words: fiber reinforced concrete, composite materials, homogenization, periodicity, finite element

1 INTRODUCTION

Since the early studies of composite materials by microscopic observation in the 1950s, recent rapid progress in computer technologies has enabled us to analyze the microscopic mechanical behaviors numerically. As a theoretical and numerical technique to solve micro-macro coupling problems, the homogenization method is becoming more and more important for the understanding of micromechanics of composite materials. This method is an applied mathematical theory first developed in the early 1980s. Guedes and Kikuchi (1990) have published a pioneering work in the engineering field of metal matrix composite material. Hassani (1998), Takano (1999) and Terada (2000) developed the homogenization technique to solve many applications of composite materials. From these works, it is recognized that the homogenization method can predict the effective elastic properties with enough accuracy for arbitrary complex microstructures

SFRC is also a composite material which can be solved by homogenization method. The short fibers are usually scattered in a random way which is difficult to model. This distribution needs to be transformed to an equivalent periodic transformation which can be analyzed by homogenization method.

This paper introduces a pattern of periodic distribution for SFRC to represent the unit cell of the structure. The

boundary conditions for the unit cell and the homogenized stiffness matrix have been derived using the homogenization method. The behavior of the composite material under flexural loading is studied before and after cracking. In the following section a brief introduction for the homogenization method is introduced.

2 HOMOGENIZATION METHOD

We assume that a two dimension of elastic body is an assembly of periodic unit cells as shown in Fig.1. Defining the macroscopic coordinate x and the microscopic one $y = x/\varepsilon$, where ε is the scale ratio and is a very small positive number. The multiscale method enables us to expand the displacements asymptotically as follows. u_i^0 and u_i^1 are macro- and microscopic displacements, respectively. In other words, macroscopic displacement u_i^0 is a kind of averaged smooth displacement. The real displacement u_i is rapidly oscillating due to the inhomogeneity from the microscopic point of view. u_i^1 is the perturbed displacement according to the microstructure. Hence, the scale factor ε is multiplied to the microscopic displacement

$$u_i = u_i(x, y) = u_i^0(x) + \varepsilon u_i^1(x, y = \frac{x}{\varepsilon}) \quad (1)$$

The equilibrium equation for an elastic problem is described as follows. D_{ijkl} is an elastic tensor and t_i is the traction applied on the boundary Γ_i . Here, the body force is neglected

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$$\int_{\Omega} D_{ijkl} \frac{\partial u_k}{\partial x_l} \frac{\partial u_i}{\partial x_j} d\Omega = \int_{\Gamma_t} t_i u_i d\Gamma \quad (2)$$

substituting equation (1) into equation (2) and taking the limit of $\varepsilon \rightarrow 0$ using the averaging principle, we get the following relation that must be satisfied over a microscopic unit cell Y

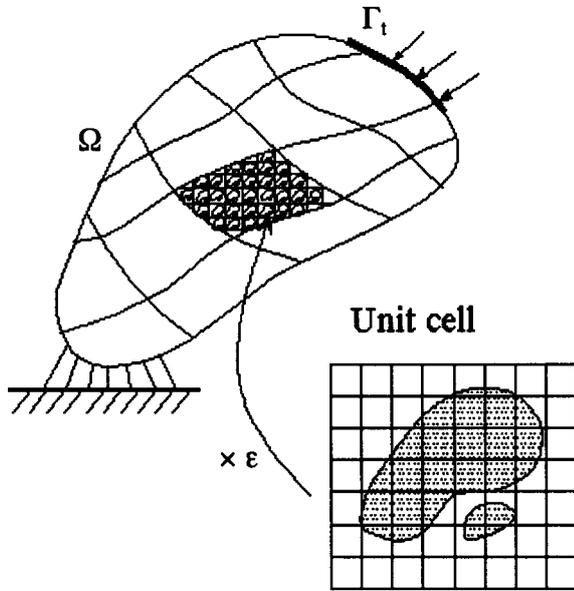


Fig.1 A macroscopic body with a microscopic unit cell

$$\int_Y D_{ijkl} \frac{\partial \chi_p^{kl}}{\partial y_p} \frac{\partial u_i}{\partial y_j} dY = \int_Y D_{ijkl} \frac{\partial u_i}{\partial y_j} dY \quad (3)$$

This equation can be solved with the periodic boundary condition. χ_p^{kl} is the characteristic displacement which is a periodic function of y . The characteristic displacement has six modes of displacement of the microstructure that reflect the mismatch of the mechanical properties of the constituents and the geometrical configuration of the constituents. The microscopic displacement depends on the macroscopic boundary conditions and the macroscopic deformation. Hence, the macroscopic strain can bridge the macroscale and microscale and the microscopic displacement can be expressed as follows

$$u_i^1 = -\chi_i^{kl}(y) \frac{\partial u_k^0}{\partial x_l} \quad (4)$$

When we use the equation (4), the existence and the uniqueness of the solution (3) are proved.

Weak forms for the homogenized macroscopic body can be derived as follows, where $|Y|$ is the volume of the microscopic unit cell. D_{ijkl}^H is the homogenized elastic

tensor that can be calculated by equation (6) after solving the microscopic equation (3)

$$\int_{\Omega} D_{ijkl}^H \frac{\partial u_k^0}{\partial x_l} \frac{\partial u_i}{\partial x_j} d\Omega = \int_{\Gamma_t} t_i u_i d\Gamma \quad (5)$$

$$D_{ijkl}^H = \frac{1}{|Y|} \int_Y (D_{ijkl} - D_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q}) dY \quad (6)$$

Finally, microscopic stresses are obtained by equation (7)

$$\sigma_{ij} = (D_{ijkl} - D_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q}) \frac{\partial u_k^0}{\partial x_l} \quad (7)$$

From these equations, the macroscopic strains, $\partial u_k^0 / \partial x_l$, bridge macro-micro behaviors. Hence, accurate calculation of macroscopic strains is essential.

Using the finite element discretization to solve the partial differential equations, the microscopic equation (3) can be written as follows

$$\left(\int_Y B^T D B dY \right) \chi^{ij} = \int_Y B^T D^{ij} dY \quad (8)$$

where D is the stress strain matrix of the constituents of the composite materials and B is the displacement-strain matrix. D^{ij} is a vector of column ij ($ij = 11, 22, 12$) of the stress-strain matrix E in 2D. χ^{ij} is the characteristic displacement vector associated with the ij mode. We define the following matrix.

$$\chi = (\chi^{11} \chi^{22} \chi^{12}) \quad (9)$$

The form in equation (8) is very similar to the well known stiffness equation

$$K \cdot \chi = f \quad (10)$$

where the stiffness matrix is written as

$$K = \int_Y B^T D B dY \quad (11)$$

and the force is written as

$$f = \int_Y B^T D^{ij} dY \quad (12)$$

Then, equation (6) can be expressed as follows

$$D^H = \frac{1}{|Y|} \int_Y D(1 - B\chi) dY \quad (13)$$

The macroscopic equation (5) can be described as follows

$$\left(\int_{\Omega} B^T D^H B d\Omega \right) \mu^0 = \int_{\Gamma_t} N^T t d\Gamma \quad (14)$$

where N is the shape function of the finite element and t is the traction vector. Microscopic stresses, equation (7), can be expressed as follows

$$\sigma = D(I - B\chi) B u^0 \quad (15)$$

Obviously, macroscopic stresses are defined by averaging the microscopic stresses. Thus equation (13) coincides with the conventional mechanics of composite materials

$$\sigma^H = D^H B u^0 \quad (16)$$

It is interesting to notice that the force vector used in

homogenization has a physical meaning and it is in fact a specific case of *initial strain loading*. To demonstrate this we recall that the nodal forces induced by initial strains are

$$f^e = \int_Y B^T D \varepsilon^0 dY \quad (17)$$

Now comparing equations (17) and (12) results in $D \varepsilon^0 = D^{\hat{ij}}$ or

$$\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11}^0 \\ \varepsilon_{22}^0 \\ \varepsilon_{12}^0 \end{Bmatrix} = \begin{Bmatrix} D_{11} \\ D_{12} \\ 0 \end{Bmatrix} \quad (18)$$

which implies that

$$\varepsilon_{11}^0 = 1, \quad \varepsilon_{22}^0 = 0 \quad \text{and} \quad \varepsilon_{12}^0 = 0 \quad (19)$$

Thus, in this case, the loading is a unit initial strain in only the y_1 direction which defines the first characteristic displacement vector χ^{11} .

For the second characteristic displacement vector χ^{22} , the load is a unit initial strain in only y_2 direction, so that

$$\varepsilon_{11}^0 = 0, \quad \varepsilon_{22}^0 = 1 \quad \text{and} \quad \varepsilon_{12}^0 = 0 \quad (20)$$

For the case of χ^{22} , the load is a unit initial shear strain loading, so that

$$\varepsilon_{11}^0 = 0, \quad \varepsilon_{22}^0 = 0 \quad \text{and} \quad \varepsilon_{12}^0 = 1 \quad (21)$$

Thus we have observed that for 2D problems, by considering three loading cases it is possible to find the homogenized stiffness matrix for the composite material.

In practice, after discretizing the domain of the unit cell, it is sufficient to run the finite element program for different unit initial strain cases. Derivation of the required boundary conditions will be discussed in the following section.

3 THE UNIT CELL OF FRC AND BOUNDARY CONDITIONS IN 2D

The short fibers are distributed randomly inside the domain as it is shown in Fig.2-a. To treat these structures with the homogenization methods, we need to convert the random distribution to periodic distribution. The random distribution almost gives the same properties of the combined material in both perpendicular directions. So we need a symmetric pattern for periodicity. Fig.2-b shows one choice of the periodic distribution of fibers. The distance between fibers is calculated according to the required volume content of fibers. The unit cell of the periodicity is shown by the hatched area in Fig.2-b and Fig.3. This kind of pattern allows similar properties of the unit cell in both directions and represents the random

distribution as the half number of fibers strengthen the horizontal direction along s_1 and the other half strengthen the vertical direction along s_2 . Each of the short fibers in the corners of the cell is the half length of the fiber with the half cross section area.

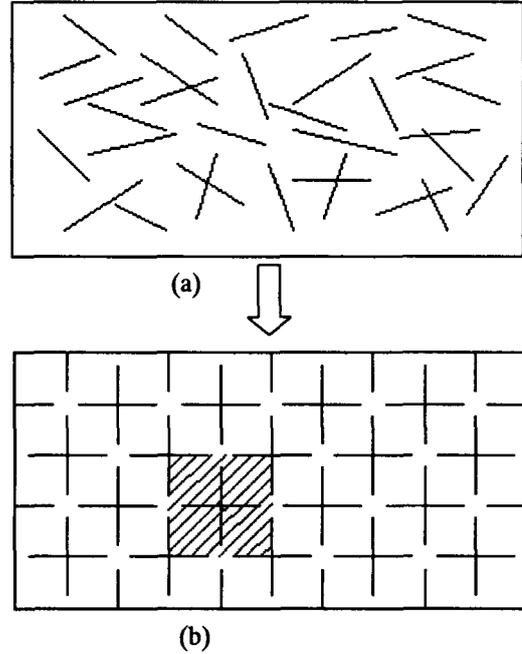


Fig.2 A macroscopic body of SFRC with random and periodic distribution

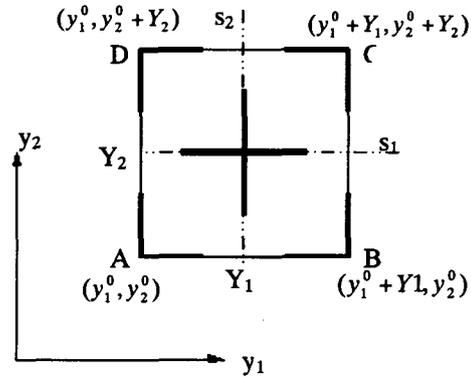


Fig.3 The unit cell of periodicity

To determine the appropriate boundary conditions, we reconsider the main assumption of homogenization theory, i.e. periodicity. The microscopic displacement field x is the Y -periodic solution of equation (10). The base cell Y is illustrated in Fig.3.

From the definition of periodicity it follows that

$$\begin{aligned} \chi(y_1^0, y_2^0) &= \chi(y_1^0 + Y_1, y_2^0) \\ &= \chi(y_1^0 + Y_1, y_2^0 + Y_2) = \chi(y_1^0, y_2^0 + Y_2) \end{aligned} \quad (22)$$

which states that the microscopic displacements of the

corner points of the base cell are equal .

Also according to the periodicity the numbers of pairs of nodes located on the opposite edges of the cell can be linked so that opposite edges have identical deformed shapes. For 2D problems the homogenization equation must be solved three times with three types of boundary conditions. The three cases are as followed

Case a

The loading to be imposed in this case is a unit initial strain in y_1 direction i.e. $\varepsilon_{11}^0 = 1$, $\varepsilon_{22}^0 = 0$ and $\varepsilon_{12}^0 = 0$.

To prevent rigid body motion, the horizontal and vertical displacements of the bottom left corner, and the vertical displacement for the bottom right corner are restricted. The loading condition and the symmetry with respect to s_2 of the base cell lead to the following boundary condition

$$u(y_1^0, y_2) = -u(y_1^0 + Y_1, y_2) \quad (23)$$

$$v(y_1^0, y_2) = v(y_1^0 + Y_1, y_2) \quad (24)$$

The previous two equations mean that the horizontal displacements of the points on the left or right edge have the same displacement of the corresponding point on the other edge but in the opposite direction. And the vertical displacements have the same value and direction.

Similarly from symmetry with respect to s_1 it follows that

$$u(y_1, y_2^0) = u(y_1, y_2^0 + Y_2) \quad (25)$$

$$v(y_1, y_2^0) = -v(y_1, y_2^0 + Y_2) \quad (26)$$

Case b

The loading condition in this case is $\varepsilon_{11}^0 = 0$, $\varepsilon_{22}^0 = 1$ and $\varepsilon_{12}^0 = 0$. As with case (a), because of the symmetry of geometry and loading, we conclude the same boundary conditions.

Case c

The loading condition in this case is a unit initial shear strain $\varepsilon_{11}^0 = 0$, $\varepsilon_{22}^0 = 1$ and $\varepsilon_{12}^0 = 0$. In this case the anti-symmetry condition exists. So the displacement of the opposite edges of the base cell maybe expressed as:

$$u(y_1^0, y_2) = u(y_1^0 + Y_1, y_2) \quad (27)$$

$$v(y_1^0, y_2) = -v(y_1^0 + Y_1, y_2) \quad (28)$$

$$u(y_1, y_2^0) = -u(y_1, y_2^0 + Y_2) \quad (29)$$

$$v(y_1, y_2^0) = v(y_1, y_2^0 + Y_2) \quad (30)$$

4 THE HOMOGENIZED STIFFNESS MATRIX

To apply the homogenization technique we need to discretize the unit cell to several elements by creating a mesh model. The fiber elements are treated like a bar

element and the concrete elements are treated as a triangle element. A regular mesh is created.

The homogenized stiffness matrix D^H is a symmetric matrix and has the form of equation (18). According to the symmetry of the pattern the values of D_{11} and D_{22} are equals. So it has three parameters which are D_{11} , D_{12} , and D_{33} . For the elastic homogenized matrix, it is easy to calculate according to the loading and boundary conditions discussed in the previous section.

In compression the concrete is assumed to be elastic so the elastic homogenized matrix can be used in all compression elements in the macros-scale domain.

In the elements subjected to tensile stress, the tensile cracks appear after the concrete reaches to the yield stress. After cracking starts the homogenized stiffness matrix changes due to the change of the material properties. The young's modulus of concrete decreases rapidly when the concrete element starts to crack. In this stage we have to estimate the young's modulus of each element of concrete after cracking. To do this, the smeared crack model for plain concrete is used. In this approach the cracked concrete is assumed to remain a continuum. Cracks are smeared over the whole area by reducing the material stiffness (Young's and shear modulus). The Young's modulus E in the direction perpendicular to the cracks direction is reduced to the value of $E(1-\omega)$ which represents the damage of the element, where ω represents the damage factor and $0 \leq \omega \leq 1$. Also, a reduced shear modulus αG is assumed on the cracked plane to account for the aggregate interlocking. The value α is a preselected constant such that $0 \leq \alpha \leq 1$. The stress strain relationship becomes as followed;

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E(1-\omega)} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{\alpha G} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} \quad (31)$$

where, ω represents the damage of concrete. The poisson's ratio $\nu=0.3$ and it is assumed to be constant before and after cracking.

The stress strain relation and ω -strain relation used in this study are derived by Bolander (1992) and S.Hiranaka (2002) as shown in Fig.4. According to their work the value of ω is calculated from equation (32)

$$\omega = \begin{cases} 0 & (\varepsilon_{11} \leq \varepsilon_t) \\ 1 - \frac{f_t}{E\varepsilon_{11}} \exp\left[-\frac{k}{\varepsilon_0} (\varepsilon_{11} - \varepsilon_t)\right] & (\varepsilon_{11} \geq \varepsilon_t) \end{cases} \quad (32)$$

where f_t and ϵ_t are the maximum elastic tensile stress and strain of concrete respectively. ϵ_{t0} is the strain when the element becomes perfectly damaged, and k is an empirical constant to express the strain softening and its average value equals 5. In this study, $f_t = 3.1$ MPa, $\epsilon_t = f_t/E$ and $\epsilon_{t0} = 0.002$.

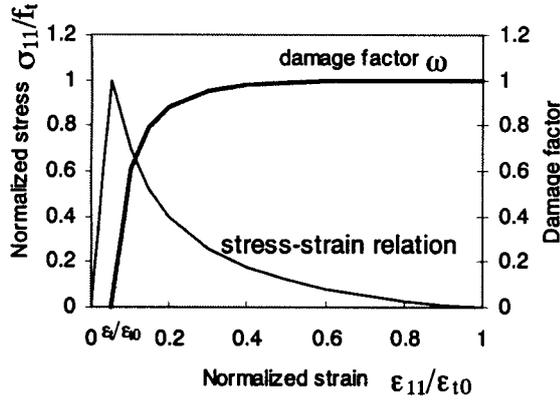


Fig.4 Stress-strain relation and ω -strain relation for plain concrete.

For the accurate calculation of the stiffness matrix, the unit cell should be solved in every integration point of the macro-scale domain in which the neighbor elements started to crack. But this way costs very much of calculation time and large memory size of computers. So to simplify the solution, the unit cell is solved to get the homogenized stiffness matrix after cracking using a uniaxial incremental displacement along s_1 . The damaged factor ω is calculated for each element and the new young's modulus which equals $E(1-\omega)$ is calculated. Then the previous homogenization technique is applied to obtain D^H . By this way we will have two homogenized matrix, one for the elements before cracking and another one for the elements after cracking.

As an application for the method D^H is estimated for the pattern shown in Fig.3. The width of the cell is 40 mm and the length of the fiber is 30mm. the fiber volume content of this example equals 2%.

Fig.5 shows the parameters of D^H with respect to the tensile strain. In the elastic range before crack the values of D_{11} , D_{12} and D_{33} are constant. These values change rapidly after cracking to another constant value. This means we can define two values for every parameter. From the figure after cracking D_{12} almost vanishes, so the value of D_{12} is set equal to zero. D_{11} and D_{22} are equals before and after cracking. D^H after cracking will be in the form:

$$D^H = \begin{bmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix} \quad (33)$$

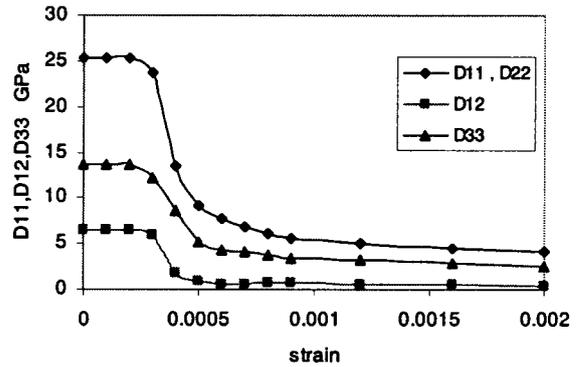


Fig.5 The values of parameters of the homogenized stiffness matrix with respect to the tensile strain

Based on these results, the stress strain relationship for $\sigma_{11}-\epsilon_{11}$ can be simplified as shown in Fig.6. After the composite element starts to crack the strength of the element decreases rapidly but it starts to recover again to carry some stress due to the existence of fibers. ϵ_t represents the tensile strain of concrete, ϵ_p represents the strain at which the fibers recover the strength of the matrix and ϵ_u is the max strain after which the element is considered totally damaged. Based on Fig.5, these values are considered as follows; $\epsilon_t=0.0003$, $\epsilon_p=0.0005$ and $\epsilon_u=0.002$.

The values of D_{11} , D_{12} and D_{33} before and after cracking should be determined by solving the unit cell using the homogenized method taking into consideration changing the dimensions of the unit cell according to the fiber content, the length, and the diameter of the fibers with keeping the shape of the pattern of the unit cell.

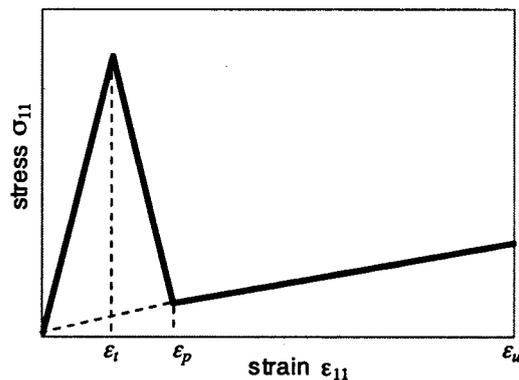


Fig.6 Stress strain relation for the composite in macrostructure

5 NUMERICAL EXAMPLE

A numerical example is solved to show the influence of short steel fibers to improve the strength of concrete in tension. A concrete specimen with dimensions 400x100x800 mm with 4-points loading is analyzed. Short steel fibers 30mm long and 0.6mm diameter are distributed with several volume contents inside the specimen. Finite element model of the macro-structure has been created with 3321 nodes and 6400 triangle elements. The microstructure model is shown in Fig 7 the width of the unit cell L is decided according to the volume fraction of fibers in the cell and it can be calculated from the following equation

$$L = \sqrt{A_f / \rho} \tag{34}$$

where A_f is the total area of fibers in the unit cell and ρ is the volume fraction of fibers.

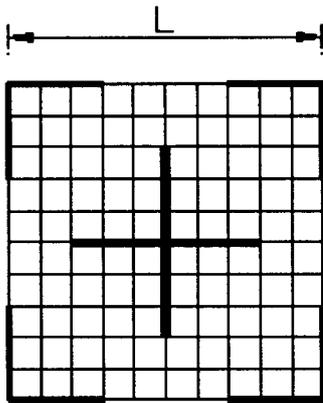


Fig.7 FE model for microstructure

In this nonlinear analysis for the macroscopic structure, the total load applied is calculated from a series of small displacement increments at the point of loading. At the completion of each incremental solution, the stiffness matrix of each element is adjusted if the element starts to damage, i.e. if the tensile strain in the element exceeds the elastic tensile strain ϵ_t .

Fig.8 shows the load deflection curve for different volume fractions of short steel fibers. It shows that the fibers do not considerably influence on the flexural strength of concrete. However, the mean feature of steel fibers is that it can increase the toughness (energy absorption) and enhance the ductility of concrete as shown from the energy absorption-strain curve in Fig.9. The enhanced behavior of steel fiber reinforced concrete over its unreinforced counter-parts comes from its improved capacity to absorb energy during fracture.

While a plain unreinforced matrix fails in a brittle manner at the occurrence of cracking stresses, the fibers in fiber reinforced concrete continue to carry stress beyond matrix cracking, which helps maintain structural integrity and cohesiveness in the material.

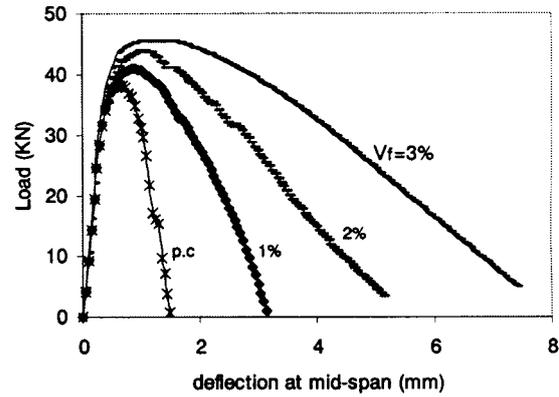


Fig.8 The load – deflection relation for SFRC with different fiber content

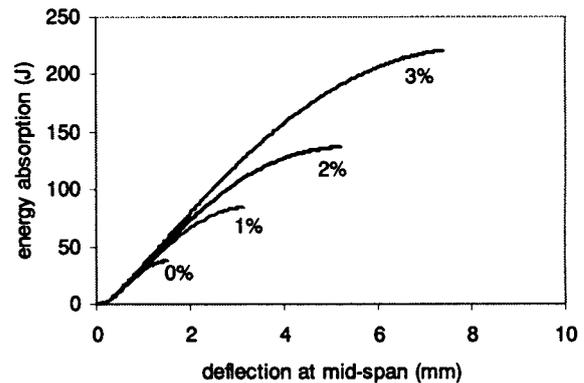


Fig.9 The energy absorption for the computed SFRC models

6 CONCLUSIONS

Homogenization method is used to model SFRC as a composite material. A suitable unit cell to imitate the properties of the random distribution of fibers is introduced. The homogenized stiffness matrix for the composite is defined before and after cracking. A numerical model is presented to illustrate the effect of fiber content on the behavior of the composite material using the homogenization technique.

For future work, extension the method to 3D with a 3D unit cell is necessary. Also the study will be extended to study the effect of the shape, length and diameter of the fibers on the behavior of SFRC.

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