ON THE COHOMOLOGIES OF FREE LOOP SPACES AND RATIONAL CYCLIC HOMOLOGIES

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PREFACE

This is a thesis submitted to the graduate school in in partial fulfillment of the requirements for the academic degree of Doctor of Science in the field of Mathematics at Okayama University.

The material in the thesis is organized by the following four chapters:

In Chapter 0, we give some basic tools. In Section 1, we define Sullivan minimal model and formal space. In Section 2, we give the Sullivan minimal models of a free loop space, LX, and the Borel space by the free circle action on LX, $ES^1 \times_{S^1} LX$. In Section 3, we explain Eilenberg-Moore spectral sequence.

In Chapter 1, we calculate the mod p cohomology algebra of LX by using an Eilenberg-Moore spectral sequence. In Section 2, we see that the E_2-term is isomorphic to the Hochschild homology of the mod p cohomology of X. In Section 3, we give a useful subalgebra of the Hochschild homology for a commutative graded complete algebra. In Section 4, we calculate the mod p cohomology algebra of LX when the mod p cohomology algebra of X is isomorphic to a truncated polynomial algebra generated by one element or an exterior algebra generated by two elements. In Section 5, we apply Section 3 to provide the lower bounds on the dimensions of a Hodge decomposition factors of the rational cohomology of LX.

In Chapter 2, we apply the method of Chapter 1 to the vanishing problem of a 3-dimensional characteristic class in string theory. In Section 1, we define string class. In Section 2, we see that a subalgebra of the Hochschild homology of the mod p cohomology of X is isomorphic to the mod p cohomology algebra of LX below degree 3. In Section 3, we give a necessary and sufficient condition for the vanishing of string class under a certain assumption on the 4-dimensional integral cohomology of X. In Section 4, we apply this result to 4-manifolds and homogeneous spaces of rank one.

In Chapter 3, we consider a rational homotopical property of $ES^1 \times_{S^1} LX$. In Section 1, we consider the normality of it. In Section 2, we give a necessary and sufficient condition for the formality of it. In Section 3, we also consider the Connes' periodicity map of rational cyclic cohomology.

I would like to express my gratitude to my supervisor Professor M.Mimura, who gave me many helpful comments for this thesis and to Professor K.Kuribayashi in Okayama University of Science, who induced me to the joint research described in Chapter 1 and Chapter 2. I also would like to thank Professors M.Fujii, K.Shimakawa and I.Yoshioka for their encouragements.

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Contents

Introduction

Let X be a path-connected and simply connected space and S^1 the circle. We denote the free loop space of X by LX, by which we mean the space of all continuous maps from S^1 into X endowed with compact-open topology. If X is a manifold, the free loop space is an interesting object in the study of the set of closed geodesics. In 1969, Gromoll and Meyer showed, for a simply connected closed Riemannian manifold M, that if there exists a field $\mathbf{k}$ such that the sequence of Betti numbers $\{b_n(LM, \mathbf{k}) = \dim_{\mathbf{k}} H_n(LM; \mathbf{k})\}_{n>0}$ is unbounded, then for any Riemannian metric there exist infinitely many geometrically distinct closed geodesics on M ([**18**]). Their statement induces us to study the (co)homology of free loop space. Let $\mathbf{Q}$ be the rational number field. If $\mathbf{k} = \mathbf{Q}$, the minimal model ([**48**],[**8**]) established by Sullivan is useful for this purpose. It is a commutative differential graded algebra (CDGA) such that its algebra is free and its differential is decomposable. In 1979, Sullivan and Vigué-Poirrier showed, by using the Sullivan minimal model of LX, that the rational cohomology algebra of X, $H^*(X; \mathbf{Q})$, requires at least two generators if and only if $\{b_n(LX, \mathbf{Q})\}_{n>0}$ is unbounded ([**49**]). We must pay attention that, for a field $\mathbf{k}$ with positive characteristic, $H^*(X; \mathbf{k})$ may be generated by at least two cohomology classes even though $H^*(X; \mathbf{Q})$ is generated by a single cohomology class. Put $r_X + 1 = \inf\{i \geq 2; H^i(X; \mathbf{k}) \neq 0\}$ and $n_X = \sup\{i; H^i(X; \mathbf{k}) \neq 0\}$. Recall that X is said $\mathbf{k}$-formal ([**8**],[**12**]) if there is a chain of weak equivalences between the singular cochain complex over $\mathbf{k}$, $C^*(X; \mathbf{k})$, and the CDGA of $H^*(X; \mathbf{k})$ with differential zero, $(H^*(X; \mathbf{k}), 0)$. In 1991, Halperin and Vigué-Poirrier showed that, under the assumption that char $\mathbf{k} \geq n_X/r_X$ or X is $\mathbf{k}$-formal, $H^*(X; \mathbf{k})$ requires at least two generators if and only if $\{b_n(LX, \mathbf{k})\}_{n>0}$ is unbounded ([**22**]). Also they conjectured that the same phenomenon should hold without the additional assumption. But it is still open.

On the other hand, there is a natural free S^1-action on LX. In 1985, Burghelea and Vigué-Poirrier constructed the Sullivan minimal model of the Borel space by the S^1-action, $ES^1 \times_{S^1} LX$ ([**7**]). Also they calculated the Poincaré series of $H^*(ES^1 \times_{S^1} LX; \mathbf{Q})$ when $H^*(X; \mathbf{Q})$ are truncated polynomial algebras generated by one generator and showed, and as an equivariant version of the result of Sullivan and Vigué-Poirrier in the first paragraph,

that $H^*(X; \mathbf{Q})$ requires at least two generators if and only if $\{b_n(ES^1 \times_{S^1} LX, \mathbf{Q})\}_{n>0}$ is unbounded.

We wish to find a possibly necessary and sufficient condition for the cohomology of X such that we can get an information for LX or $ES^1 \times_{S^1} LX$. First, we calculate the cohomology algebras of LX under certain additional assumptions for them of X. But we remark that each result is false in the consideration of degrees if any part of the assumption is broken off. Especially, we need some conditions for Steenrod operations on the cohomology of X in a case. Second, we consider a problem in string theory where the objects are the 4-dimensional integral cohomology of X and the 3-dimensional integral cohomology of LX. Our algebraic method exists in the homological algebra over the residue class field of p. Therefore we assume a sufficient, but possibly general, condition on the integral cohomology of X so that the mod p reduction doesn't loose useful informations. Finally, we give a necessary and sufficient condition for the $\mathbf{Q}$-formality of $ES^1 \times_{S^1} LX$.

This thesis is divided into the following three chapters:

In Chapter 1, we consider the algebra structure of the mod p cohomology of LX, $H^*(LX; \mathbf{Z}/p)$, where p is a prime number or zero. Here $\mathbf{Z}/p$ means the residue class field of p if $p \neq 0$ and the rational number field $\mathbf{Q}$ if $p = 0$. For calculating $H^*(LX; \mathbf{Z}/p)$ from $H^*(X; \mathbf{Z}/p)$, we use the Eilenberg-Moore spectral sequence ([**11**]) for the following fiber square:

$$\begin{array}{ccc} LX & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ X & \underset{\Delta}{\longrightarrow} & X \times X, \end{array}$$

where Δ is the diagonal map. In the procedure, the Hochschild homology of the CDGA $(H^*(X; \mathbf{Z}/p), 0)$:

$$HH_*(H^*(X; \mathbf{Z}/p), 0) = Tor_{H^*(X;\mathbf{Z}/p) \otimes H^*(X;\mathbf{Z}/p)}(H^*(X; \mathbf{Z}/p), H^*(X; \mathbf{Z}/p))$$

appears as the E_2-term of the spectral sequence. In 1980, L.Smith indicated that there is a commutative differential graded algebra (called Koszul-Tate complex) whose cohomology is isomorphic to $HH_*(\Lambda)$ if Λ is a commutative graded complete intersection algebra

(GCI-algebra) over $\mathbf{Q}$ ([**44**]). We consider this complex in the case that $p \neq 0$ ([**25**]), and calculate $H^*(LX;\mathbf{Z}/p)$, as a special case of GCI-algebras, when $H^*(X;\mathbf{Z}/p)$ are some truncated polynomial algebras generated by one element or some exterior algebras generated by two elements under some conditions on the degree(s) of the generator(s) and the positive characteristic p. By arguments based on total degrees and filtration degrees of elements, we see that the spectral sequence collapses at the E_2-term and we can solve all extension problems. It is known that the Eilenberg-Moore spectral sequence collapses at E_2-term if X is $\mathbf{Z}/p$-formal or if $H^*(X;\mathbf{Z}/2)$ satisfies some conditions on the Steenrod operations ([**46**]). We also consider a part of them in which we can solve all extension problems by degree arguments. In general, for the rational de Rham complex $(\Omega^*(X),\partial)$ of X, $HH_*(\Omega^*(X),\partial)$ is isomorphic to $H^*(LX;\mathbf{Q})$ as an algebra since the CDGA is commutative. Furthermore, by virtue of the $\mathbf{Q}$-formality of X, there is a chain of isomorphisms:

$$HH_*(H^*(X;\mathbf{Q}),0) \overset{\cong}{\leftarrow} HH_*(\mathcal{M}(X)) \overset{\cong}{\rightarrow} HH_*(\Omega^*(X),\partial) \overset{\cong}{\rightarrow} H^*(LX;\mathbf{Q})$$

as algebras if $H^*(X;\mathbf{Q})$ is a GCI-algebra ([**44**]). Here $\mathcal{M}(X)$ means the Sullivan minimal model of X. This means that not only the Eilenberg-Moore spectral sequence collapses at E_2-term, but also all extension problems is already solved. Thus there is a big distance between '$p \neq 0$' and '$p = 0$' in the calculation of $H^*(LX;\mathbf{Z}/p)$. Anyway, it is confirmed that the conjecture of Halperin and Vigué-Poirrier in the first paragraph is true at least in our assumptions where $H^*(X;\mathbf{Z}/p)$ is generated by two elements. In Section 5, we apply the consideration for Hochschild homology of above to provide the lower bounds on the dimensions of a rational Hodge decomposition factors of $H^*(LX;\mathbf{Q})$ ([**5**],[**7**]).

In Chapter 2, we apply the method of Chapter 1 to the vanishing problem of a 3-dimensional characteristic class in a string theory ([**3**],[**36**]). Let M be a simply connected differential manifold and LM a the free loop space of all smooth maps from S^1 into M. Let $\mathbf{Z}$ be the integer ring, $\int_{S^1} : H^4(S^1 \times LM;\mathbf{Z}) \to H^3(LM;\mathbf{Z})$ the integration map along S^1 and $ev : S^1 \times LM \to M$ the evaluation map. Then the map

$$\mathcal{D}_M = \int_{S^1} \circ\, ev^* \ :\ H^4(M;\mathbf{Z}) \to H^3(LM;\mathbf{Z})$$

is a derivation ([**27**]). Let ξ be an $SO(n)$-bundle with a spin structure $Spin(n) \to Q \to M$ for $n \geq 5$. In 1992, McLaughlin defined the string class $\mu(Q)$ in $H^3(LM;\mathbf{Z})$, as an

obstruction to lift the structure group of the $LSpin(n)$-bundle $LQ \to LM$ to $\widehat{LSpin(n)}$. Here $S^1 \to \widehat{LSpin(n)} \to LSpin(n)$ is the universal central extension. Let $p_1(\xi)$ be the first Pontrjagin class of ξ, which is an element of $H^4(X;\mathbf{Q})$. Then he showed that $\mathcal{D}_M$ carries $\frac{1}{2}p_1(\xi)$ to the string class $\mu(Q)$([**36**]). This say that, when $\mathcal{D}_M$ is injective, $\mu(Q)$ vanishes if and only if $\frac{1}{2}p_1(\xi)$ vanishes. Furthermore, he showed $\mathcal{D}_M$ is injective if M is 2-connected. In 1996, Kuribayashi showed that $\mathcal{D}_M$ is injective if $H^4(M;\mathbf{Z})$ is torsion free and rank $H^2(M;\mathbf{Z}) \leq 1$ ([**26**]). We generalize this statement by considering a subalgebra of $H^*(LM;\mathbf{Z}/p)$ isomorphic to a subalgebra of $HH_*(H^*(M;\mathbf{Z}/p))$ below degree 3. We show, under the assumption: $H^4(M;\mathbf{Z}) \cong \mathbf{Z} \oplus \cdots \oplus \mathbf{Z} \oplus \mathbf{Z}/p_1 \oplus \cdots \oplus \mathbf{Z}/p_k$ where p_i is prime for any i and $x^2 = 0$ for any element $x \in H^2(M;\mathbf{Z}/2)$ if $H^4(M;\mathbf{Z})$ has 2-torsion, that $\mathcal{D}_M$ is injective. In Section 4, we apply this result to 4-manifolds and homogeneous spaces of rank one by referring the table of [**35**].

In Chapter 3, we consider a rational homotopical property of $ES^1 \times_{S^1} LX$. We remark that the algebra structures of $H^*(LX;\mathbf{Q})$ and $H^*(ES^1 \times_{S^1} LX;\mathbf{Q})$ are very complicated. These difficulties are made up by a rich structure of Massey products or higher order cup-products, which are caused since LX and $ES^1 \times_{S^1} LX$ are not $\mathbf{Q}$-formal in general. All these rational homotopical informations are carried by the Sullivan minimal models which are simpler to describe for these spaces than the rational cohomology algebras. Let X be not rationally contractible and $H^*(X;\mathbf{Q})$ finitely generated. In 1998, Dupont and Vigué-Poirrier showed that LX is $\mathbf{Q}$-formal if and only if $H^*(X;\mathbf{Q})$ is free as an algebra ([**9**]). We show, as an equivariant version of it, that $ES^1 \times_{S^1} LX$ is $\mathbf{Q}$-formal if and only if $H^*(X;\mathbf{Q})$ is an exterior algebra generated by only one element with degree odd, i.e., X has the rational homotopy type of an odd dimensional sphere. By the way, it is known that $H^*(ES^1 \times_{S^1} LX;\mathbf{Q})$ is isomorphic to the rational cyclic cohomology of X, $HC^*(X;\mathbf{Q})$ ([**4**], [**6**], [**7**]). In Section 3, we consider the triviality of the Connes' periodicity map restricted on a factor of $HC^*(X;\mathbf{Q})$.

CHAPTER 0

Preliminaries

Before we proceed to the main subjects, we introduce two fundamental tools: "Sullivan minimal model" and "Eilenberg-Moore spectral sequence" in this chapter.

1. Sullivan minimal model and formality

Let $\mathbf{k}_0$ be a field of characteristic zero. A graded $\mathbf{k}_0$-algebra $A = \oplus_{i \geq 0} A^i$ is said to be *commutative* if $xy = (-1)^{ij} yx$ for $x \in A^i$ and $y \in A^j$. Here A^i is the subspace of A whose elements are homogeneous of degree i. A $\mathbf{k}_0$-commutative graded algebra A is called a $\mathbf{k}_0$-*commutative differential graded algebra* ($\mathbf{k}_0$-CDGA for short) if it has a differential d_A of degree 1. We denote it by (A, d_A). An element a of A is called a d_A-*cocycle* if $d_A(a) = 0$ or is said to be *closed* and is said to be d_A-*exact* if there exists an element b of A such that $d(b) = a$. It is said *connected* if $H_0(A, d_A) = \mathbf{k}_0$. An $\mathbf{k}_0$-algebra homomorphism of degree zero $\phi : (A, d_A) \to (B, d_B)$ is called a *CDGA-morphism* if it commutes with differentials: $\phi \circ d_A = d_B \circ \phi$. In the following, we denote $A \otimes_{\mathbf{k}} B$ as simply $A \otimes B$ if both A and B are algebras over a field $\mathbf{k}$, unless we explicitly mention otherwise. We denote $\wedge Z$ as the free algebra over a $\mathbf{k}_0$-graded vector space $Z = \oplus_{i \geq 1} Z^i$ (Z^i is the subspace of Z whose elements are of degree i), that is,

$$\wedge Z = (\text{ the } \mathbf{k}_0\text{-polynomial algebra over } Z^{even}) \otimes (\text{ the } \mathbf{k}_0\text{-exterior algebra over } Z^{odd}),$$

where $Z^{even} = \oplus_{i \geq 1} Z^{2i}$ and $Z^{odd} = \oplus_{i \geq 0} Z^{2i+1}$. Then $\wedge(\{z_i\}_{i \in I}) = \wedge Z$ if $\{z_i\}_{i \in I}$ is a basis of Z. Let $\wedge^i Z$ denote the subspace of $\wedge Z$ generated by the products of i elements of Z. Put $\wedge^+ Z = \oplus_{i \geq 1} \wedge^i Z$; this is the ideal of $\wedge Z$ generated by a basis of Z.

Definition 1.1 ([**8**, Section 1], [**20**, Part II]). *A free* $\mathbf{k}_0$-*CDGA* $\mathcal{M} = (\wedge Z, d)$ *is the* minimal model *of a connected* $\mathbf{k}_0$-*CDGA* $\mathcal{A} = (A, d_A)$ *if*

1. *the differential d is nilpotent, i.e., Z has a well ordered basis $\{z_i\}_{i\in I}$ with $d(z_i) \in \wedge(Z_{<i}) = \wedge(\{z_j\}_{j<i})$,*
2. *the differential d is decomposable, i.e., $d(z) \in \wedge^+Z \cdot \wedge^+Z$ for any $z \in Z$,*
3. *there exists a $\mathbf{k}_0$-CDGA-morphism $\rho : \mathcal{M} \to \mathcal{A}$ which induces an $\mathbf{k}_0$-algebra isomorphism $\rho_* : H_*(\mathcal{M}) \cong H_*(\mathcal{A})$.*

The minimal model exists, for any $\mathbf{k}_0$-CDGA $\mathcal{A}$, uniquely up to $\mathbf{k}_0$-CDGA-isomorphism ([**19**, chapter 6]). If $\psi : \mathcal{A} \to \mathcal{B}$ is a quasi-isomorphism, i.e., $\psi_* : H_*(\mathcal{A}) \cong H_*(\mathcal{B})$, there is a quasi-isomorphism $\mathcal{M}(\mathcal{A}) \to \mathcal{B}$ by the obstruction theory of $\mathbf{k}_0$-CDGA ([**17**]). So the minimal model of $\mathcal{A}$ is denoted by $\mathcal{M}(\mathcal{A})$. Let $(\Omega^*(X), \partial)$ be the rational de Rham complex ([**17**, p.183]) of a path connected space X, namely, the $\mathbf{Q}$-CDGA of the $\mathbf{Q}$-polynomial forms on X with exterior differentials. There is an $\mathbf{Q}$-graded algebra isomorphism $H_*((\Omega^*(X), \partial)) \cong H^*(X; \mathbf{Q})$ induced by the integration chain map from $(\Omega^*(X), \partial)$ to the singular cochain complex of X with $\mathbf{Q}$-coefficient, $(C^*(X; \mathbf{Q}), d)$ ([**17**, p.183]). We remark that, for (differential) manifold M and the ordinary de Rham comlpex $(A^*(M), d)$, there is a chain of CDGA-morphisms between $(\Omega^*(X), \partial) \otimes \mathbf{R}$ and $(A^*(M), d)$ which induces $H_*((\Omega^*(X), \partial) \otimes \mathbf{R}) \cong H_*(A^*(M), d) \cong H^*(X; \mathbf{R})$ as algebras ([**17**, Appendix,2]). The minimal model of the $\mathbf{Q}$-CDGA $(\Omega^*(X), \partial)$ is called the *Sullivan minimal model of* X, denoted by $\mathcal{M}(X)$.

For a group G, the lower central series of G is

$$\cdots \subset \Gamma^{i+1}(G) \subset \Gamma^i(G) \subset \Gamma^{i-1}(G) \subset \cdots \subset \Gamma^1(G),$$

by setting $\Gamma^1(G) = G$, $\Gamma^{i+1}(G) = [G, \Gamma^i(G)]$ for $i \geq 1$. Recall that a group G is called *nilpotent* if $\Gamma^j(G) = 1$ for sufficiently large. Let $w : G \to Aut(H)$ be an action of G on an abelian group H. The lower central w-series of G is

$$\cdots \subset \Gamma_w^{i+1}(H) \subset \Gamma_w^i(H) \subset \Gamma_w^{i-1}(H) \subset \cdots \subset \Gamma_w^1(H),$$

by setting $\Gamma_w^1(H) = H$, $\Gamma_w^{i+1}(H) =$ the group generated by $\{gh - hg\ ; g \in G, h \in \Gamma_w^i(H)$ for $i \geq 1$. We say *the action w of G on an abelian group H is nilpotent* if if $\Gamma_w^j(G) = 1$ for sufficiently large. A space X is called *nilpotent* ([**23**]) if

1. $\pi_1(X)$ is nilpotent group and

2. the action of $\pi_1(X)$ on $\pi_n(X)$ is nilpotent for any $n > 1$.

Suppose X is a path connected and nilpotent space. Then it is known ([**8**, (3.3)]) that $\mathcal{M}(X)$ determines the rational homotopy type of X as follows:

$$X_{(0)} \simeq Y_{(0)} \text{ if and only if } \mathcal{M}(X) \cong \mathcal{M}(Y) \text{ as CDGAs} \tag{1.1}$$

for a path connected and nilpotent space Y, where "$\simeq$" means a homotopy equivalence, and $X_{(0)}$ is the $\mathbf{Q}$-localization of X ([**23**]). Especially, if X is simply connected, it is known ([**8**, (3.3)]) that

$$Z^i \cong \mathrm{Hom}(\pi_i(X), \mathbf{Q}) \text{ if } \mathcal{M}(X) \text{ is given by } (\wedge Z, d) \tag{1.2}$$

for any i.

Let $(H^*(X; \mathbf{k}_0), 0)$ be the $\mathbf{k}_0$-CDGA equipped with the differential $d = 0$ on the commutative graded $\mathbf{k}_0$-algebra $H^*(X; \mathbf{k}_0)$. In general, $\mathcal{M}(X) \otimes_{\mathbf{Q}} \mathbf{k}_0$ is not isomorphic to the minimal model of $(H^*(X; \mathbf{k}_0), 0)$. For example, one see in [**16**, p.486] that the (real) minimal model $\mathcal{M}(X) \otimes_{\mathbf{Q}} \mathbf{R}$ of the homogeneous space $X = SU(6)/SU(3) \times SU(3)$, for the special unitary group $SU(n)$, is different from that of the real cohomology of X, $(H^*(X; \mathbf{R}), 0)$. In fact, the minimal model is given by

$$\mathcal{M}(X) \otimes_{\mathbf{Q}} \mathbf{R} \cong (\wedge(x_4, x_6, y_7, y_9, y_{11}), d),$$

where $d(x_4) = 0, d(x_6) = 0, d(y_7) = x_4^2, d(y_9) = x_4x_6, d(y_{11}) = x_6^2$ with $\deg x_i = i, \deg y_i = i$ ([**16**, p.487]). Also [**16**, p.488] says the following: $H^p(X; \mathbf{R}) = 0$ for $p > 19$, $H^*(X; \mathbf{R})$ is generated by four elements $\{\alpha_4, \alpha_6, \alpha_{13}, \alpha_{15}\}$ as an algebra, and for any element α_i of $H^*(X; \mathbf{R})$, there are the relations: $\alpha_i \cdot \alpha_j = 0$ if $i + j \neq 19$ and $\alpha_i \cdot \alpha_j = \alpha_{19}$ if $i + j = 19$. Then we can conclude that

$$\begin{aligned} H^*(X; \mathbf{R}) &\cong \frac{\mathbf{R}[x_4, x_6] \otimes \Lambda_{\mathbf{R}}(w_{13}, w_{15})}{(x_4^2, x_4x_6, x_6^2, x_4w_{13}, x_6w_{15}, w_{13}w_{15}, x_4w_{15} - x_6w_{13})} \\ &\cong H^*\left((S^4 \times S^{15}) \sharp (S^6 \times S^{13}); \mathbf{R}\right) \end{aligned}$$

as algebras. Here '$X\sharp Y$' means the connected sum of X and Y and $\Lambda_{\mathbf{R}}(w)$ means the $\mathbf{R}$-exterior algebra over w. Then, for example, there must exist an indecomposable element u of degree 16 such that $d(u) = x_4w_{13}$ in the minimal model of $(H^*(X; \mathbf{R}), 0)$. Thus we see that it is different from $\mathcal{M}(X) \otimes_{\mathbf{Q}} \mathbf{R}$.

DEFINITION 1.2 ([**8**, Section 4]). *A CDGA $\mathcal{A}$ is called* $\mathbf{k}_0$*-formal if* $\mathcal{M}(\mathcal{A})$ *is* $\mathbf{k}_0$*-CDGA-isomorphic to the minimal model of* $(H^*(\mathcal{A}), 0)$.

If $\mathcal{A}$ is formal, there is a CDGA-morphism $f : \mathcal{M}(\mathcal{A}) \to (H_*(\mathcal{A}), 0)$ such that $f_* : H_*(\mathcal{M}(\mathcal{A})) \cong H_*(H_*(\mathcal{A}), 0) = H_*(\mathcal{A})$ ([**17**]). For example, the $\mathbf{k}_0$-minimal model $\mathcal{M} = (\wedge(x, y, z), d)$, where $|x| = 3, |y| = 5, |z| = 7$ and $d(x) = d(y) = 0$, $d(z) = xy$ is not formal. In fact, if it is formal, there must be a quasi-isomorphism $f : \mathcal{M} \to (H_*(\mathcal{M}), 0)$. Then we see that $f(x) = x, f(y) = y, f(z) = 0$ by a degree argument since $H_*(\mathcal{M})$ does not contain any element of degree 7. Since $d(xz) = xxy = 0$ and xz can not be d-exact also by a degree argument, the element xz represents a non-zero element of $H_*(\mathcal{M})$, i.e., $[xz] \neq 0$. Since $f_*([xz]) = [f(xz)] = 0$, it contradicts the injectivity of f_*.

THEOREM 1.3 ([**48**, Theorem 12.1],[**21**, Corollary 6.9]). *The notion of formality is independent of the ground field* $\mathbf{k}_0$*, especially* $\mathcal{M}(X) \otimes_{\mathbf{Q}} \mathbf{k}_0$ *is* $\mathbf{k}_0$*-formal if and only if* $\mathcal{M}(X)$ *is* $\mathbf{Q}$*-formal.*

So we say simply $\mathbf{k}_0$-CDGA (rationally) *formal* if it is $\mathbf{k}_0$-formal. A path connected and nilpotent space X is said *formal* if the rational de Rham complex $(\Omega^*(X), \partial)$ is formal. For example, spheres, projective spaces, and Hopf spaces are formal. The product space or bouquet of formal spaces is formal ([**21**, Lemma 1.6]). If $H^p(X; \mathbf{k}_0) = 0$ unless $p = 0$ or $l < p < 3l + 2$ $(l > 1)$, X is formal ([**21**, Corollary 5.16]). If $H^*(X; \mathbf{k}_0) \cong \mathbf{k}_0[x_1, .., x_n]/(\rho_1, .., \rho_n)$ as an algebra, where $\rho_1, .., \rho_n$ is a regular sequence, X is formal ([**21**]). It is known that a wide class of homogeneous spaces are formal ([**16**]). But we have seen that $SU(6)/SU(3) \times SU(3)$ is not formal. Also it is known that symmetric spaces and compact Kähler manifolds are formal ([**16**, X] and [**8**, Section 6], respectively). There is an interesting conjecture; "Any compact simply connected symplectic manifold is formal."([**31**], [**50**, p.198]).

2. The models of LX and $ES^1 \times_{S^1} LX$

Let X be a path-connected simply connected space. This section is discussed some CDGAs over $\mathbf{k}_0 = \mathbf{Q}$.

DEFINITION 2.1. *The space of all continuous maps from the circle* S^1 *into* X*,* $LX := Map(S^1, X)$*, endowed with compact-open topology is said free loop space of* X.

There exists an S^1-action $\mu : S^1 \times LX \to LX$ defined by $\mu(\theta^{-1}, f) = f \circ T_\theta$, where $f \in LX$ and T_θ is the translation $T_\theta : S^1 \to S^1$ corresponding to $\theta \in S^1$. Let ES^1 be a contractible space with a free S^1-action.

DEFINITION 2.2. *The Borel space* $ES^1 \times_{S^1} LX$ *is the quotient of the product* $ES^1 \times LX$ *by the equivalence relation* $(y, \mu(\theta^{-1}, f)) \sim (\theta \cdot y, f)$ *for any* $y \in ES^1$*, any* $f \in LX$ *and any* $\theta \in S^1$.

We know that if X is path-connected and simply connected, then LX and $ES^1 \times_{S^1} LX$ are connected and nilpotent ([**23**, Theorem 2.5] and [**51**, p.41] respectively). Following [**49**], we define a CDGA $(\wedge\overline{Z} \otimes \wedge Z, \delta)$ as follows: Let $\overline{Z}^i = Z^{i+1}$ and $\overline{Z} = \oplus_{i \geq 1} \overline{Z}^i$. Define $\beta : \wedge\overline{Z} \otimes \wedge Z \to \wedge\overline{Z} \otimes \wedge Z$ to be the unique derivation of degree -1 with the property:

$$\beta(z) = \overline{z} \text{ for all } z \in Z \text{ and } \beta(\overline{z}) = 0 \text{ for all } \overline{z} \in \overline{Z}.$$

(Here a linear map β is called a *derivation* if $\beta(uv) = \beta(u)v + (-1)^{deg(u)}u\beta(v)$ for any $u, v \in \wedge\overline{Z} \otimes \wedge Z$.) Then it is easy to check that $\beta \circ \beta = 0$. In fact, $(\beta \circ \beta)(uv) = \beta(\beta(u)v + (-1)^{deg(u)}u\beta(v)) = (-1)^{deg(u)-1}\beta(u)\beta(v) + (-1)^{deg(u)}\beta(u)\beta(v) = 0$. Here recall ([**7**, p.251-252]) that

$$(\wedge\overline{Z} \otimes \wedge Z, \beta) \text{ is acyclic and especially } \beta\mid_{\wedge Z} \text{ is an injection.} \tag{2.1}$$

Now define $\delta : \wedge\overline{Z} \otimes \wedge Z \to \wedge\overline{Z} \otimes \wedge Z$ as the unique derivation of degree $+1$ with the property:

$$\delta(z) = d(z) \text{ for all } z \in Z \text{ and } \delta(\overline{z}) = -\beta(d(z)) \text{ for all } \overline{z} \in \overline{Z}.$$

Then it is also easy to check that $\delta \circ \delta = 0$ and $\delta \circ \beta + \beta \circ \delta = 0$. The the minimal model of LX can be expressed in terms of that of X as follows:

THEOREM 2.3 ([**49**, Theorem]). *The minimal model of* LX *is given by* $(\wedge\overline{Z} \otimes \wedge Z, \delta)$.

Let t be an element of degree 2 $\mathbf{Q}[t]$ the polynomial algebra generated by t. By the definition of free algebra, we have $\wedge(t) = \mathbf{Q}[t]$ Following [**7**], we define a CDGA $(\mathbf{Q}[t] \otimes$

$\wedge\overline{Z}\otimes\wedge Z, D)$ as follows: Define $D : \mathbf{Q}[t]\otimes\wedge\overline{Z}\otimes\wedge Z \to \mathbf{Q}[t]\otimes\wedge\overline{Z}\otimes\wedge Z$ as the unique derivation of degree $+1$ with the property:

$$D(t) = 0 \text{ and } D(u) = \delta(u) + t\cdot\beta(u) \text{ for all } u \in \wedge\overline{Z}\otimes\wedge Z.$$

Then it holds that $D\circ D = 0$. Then the minimal model of $ES^1\times_{S^1} LX$ can be expressed in terms of that of X as follows:

THEOREM 2.4 ([**7**, Theorem A]). *The minimal model of $ES^1\times_{S^1} LX$ is given by $(\mathbf{Q}[t]\otimes\wedge\overline{Z}\otimes\wedge Z, D)$.*

REMARK 2.5. *For a fibration: $F\to E\to B$ over a simply connected space B, there is a Koszul-Sullivan(KS-)extension* ([**19**, p.14])*:*

$$(\Omega^*(B),\partial)\to(\Omega^*(B)\otimes\wedge V, D)\to(\wedge V,\overline{D}),$$

where a Koszul-Sullivan basis $\{v_i\}_{i\in I}$, i.e., well-ordered basis $i\in I$ of V such that, for each $i\in I$, $D(v_i)\in B^*\otimes\wedge(V_{<i})$ and $i<j$ if $|v_i|<|v_j|$. Here $V_{<i}$ denotes the subspace of V generated by basis elements $\{v_j; j<i\}$ such that

$$\begin{array}{ccccc}
(\Omega^*(B),\partial) & \xrightarrow{incl.} & (\Omega^*(B)\otimes\wedge V, D) & \xrightarrow{proj.} & (\wedge V,\overline{D}) \\
\downarrow{=} & & \downarrow{\rho} & & \downarrow{\overline{\rho}} \\
(\Omega^*(B),\partial) & \xrightarrow[i^*]{} & (\Omega^*(E),\partial) & \xrightarrow[p^*]{} & (\Omega^*(F),\partial)
\end{array} \tag{2.2}$$

*where ρ and $\overline{\rho}$ are quasi-isomorphisms and $(\wedge V,\overline{D})$ is the Sullivan minimal model of F ([**19**]).*

For the free loop fibration $\Omega X\xrightarrow{i} LX\xrightarrow{\pi} X$, we have the KS-extension:

$$\begin{array}{ccccc}
(\wedge Z, d) & \xrightarrow{incl.} & (\wedge Z\otimes\wedge\overline{Z},\delta) & \xrightarrow{proj.} & (\wedge\overline{Z}, 0) \\
\downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} \\
(\Omega^*(X),\partial) & \xrightarrow[\pi^*]{} & (\Omega^*(LX),\partial) & \xrightarrow[i^*]{} & (\Omega^*(\Omega X),\partial)
\end{array} \tag{2.3}$$

and for the Borel fibration $LX\xrightarrow{j} ES^1\times_{S^1} LX\xrightarrow{p} BS^1$, we have the KS-extension:

$$\begin{array}{ccccc}
(\mathbf{Q}[t], 0) & \xrightarrow{incl.} & (\mathbf{Q}[t]\otimes\wedge Z\otimes\wedge\overline{Z},\delta) & \xrightarrow{proj.} & (\wedge Z\otimes\wedge\overline{Z},\delta), \\
\downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} \\
(\Omega^*(BS^1),\partial) & \xrightarrow[p^*]{} & (\Omega^*(ES^1\times_{S^1} LX),\partial) & \xrightarrow[j^*]{} & (\Omega^*(LX),\partial),
\end{array} \tag{2.4}$$

where '$\simeq$' means quasi-isomorphic.

3. Eilenberg-Moore spectral sequence

This section is discussed over a field $\mathbf{k}$. Let p be a prime number or zero. In the following, $\mathbf{Z}/p$ means the residue class field of p if p is a prime number and it means the rational number field $\mathbf{Q}$ if $p = 0$.

Let $F\to E\xrightarrow{p} B$ be a fibration in which the base B is a connected space with $\pi_1(B)$ acting trivially on the cohomology of the fiber F. Let the following diagram be the pull back of the fibration by a continuous map $f : X\to B$:

$$\begin{array}{ccc}
E_f & \xrightarrow{\tilde{f}} & E \\
\downarrow{p'} & & \downarrow{p} \\
X & \xrightarrow[f]{} & B
\end{array} \tag{3.1}$$

By applying the singular cochain complex $C^*(\ ) = C^*(\ ;\mathbf{k})$ over a field $\mathbf{k}$ to (5.2), we get the commutative diagram;

$$\begin{array}{ccc}
C^*(E_f) & \xleftarrow{\tilde{f}^*} & C^*(E) \\
\uparrow{p'^*} & & \uparrow{p^*} \\
C^*(X) & \xleftarrow[f^*]{} & C^*(B).
\end{array} \tag{3.2}$$

This diagram of differential graded algebras allows us to endow $C^*(E)$ and $C^*(X)$ with $C^*(B)$-module structures. The map

$$\alpha : C^*(X)\otimes_k C^*(E)\to C^*(E_f)$$

is given by the composite

$$C^*(X)\otimes_k C^*(E)\xrightarrow{p'^*\otimes\tilde{f}^*} C^*(E_f)\otimes_k C^*(E_f)\xrightarrow{\cup} C^*(E_f).$$

Then α induces

$$\overline{\alpha} : C^*(X)\otimes_{C^*(B)} C^*(E)\to C^*(E_f)$$

([**34**, p.33]). Suppose $Q^* \xrightarrow{e} C^*(X)$ is a proper projective resolution of $C^*(X)$ as a right $C^*(B)$-module. If we apply the 'Tot' functor and project off the obvious factor, e induces a mapping, $e : Tot(Q^*) \to C^*(X)$. Then $\overline{\alpha}$ lifts to

$$\theta : Tot(Q^*) \otimes_{C^*(B)} C^*(E) \to C^*(E_f).$$

We define $Tor_{C^*(B)}(C^*(X), C^*(E))$ by $H_*(Tot(Q^*) \otimes_{C^*(B)} C^*(E))$.

PROPOSITION 3.1 ([**34**, 7.10,7.13]). *The map, θ, constructed above, induces an isomorphism on homology,*

$$\theta_* : Tor_{C^*(B)}(C^*(X), C^*(E)) \cong H^*(E_f; \mathbf{k})$$

as algebras.

Recall that $Tot(Q^*)$ is filtered as

$$\{0\} = F^1 \subset F^0 \subset F^{-1} \subset F^{-2} \subset \cdots \subset Tot(Q^*),$$

which is defined by

$$(F^{-n})^r = \oplus_{i+j=r, i \geq -n} (Q^i)^j,$$

where $Q^i = \oplus_j (Q^i)^j$ is the i-th complex of the projective resolution with its differential $d : (Q^i)^j \to (Q^i)^{j+1}$.

In general, let (C, d) be a filtered complex with deg $d = +1$, i.e., F^pC be a filtration of a complex (C, d) with $\cdots \subset F^{p+1}C \subset F^pC \subset F^{p-1}C \subset \cdots \subset C$ and $d(F^pC) \subset F^pC$. Then it determines a spectral sequence and if the filtration is bounded, then the spectral sequence determines $H_*(C, d)$. The filtration $\{F^p\}$ is called *exhaustive* if $C = \cup_p F^pC$ and *weakly convergent* if $F^pC \cap \text{Ker } d = \cap_r (F^pC \cap d^{-1}(F^{p+r}C))$ for any p.

LEMMA 3.2 ([**34**, Theorem 2.7]). *If (C, d) is a filtered complex such that the filter is exhaustive and weakly convergent. Then there is a spectral sequence with $E_1^{p,q} \cong H_{p+q}(F^pC/F^{p+1}C)$, which converges to $H^*(C, d)$. For any p, q, we have $E_\infty^{p,q} = E_r^{p,q}$ for large r (depending on p, q) and $E_2^{p,q} \Rightarrow H_*(C, d)$.*

By apply this lemma, we have the Eilenberg-Moore spectral sequence for the fibration (3.1).

THEOREM 3.3 ([**11**], [**42**], [**44**], [**34**, 7.1,7.14]). *There is a spectral sequence with $E_2^{p,q} = Tor^{p,q}_{H^*(B;\mathbf{k})}(H^*(X; \mathbf{k}), H^*(E; \mathbf{k}))$ which converges to $H^*(E_f, ; \mathbf{k})$ as an algebra.*

Let X be a simply connected space. In order to calculate the mod p cohomology $H^*(LX; \mathbf{Z}/p)$ from $H^*(X; \mathbf{Z}/p)$, we apply Theorem 3.3 for the fiber square in the following.

Let $X^I = Map([0,1], X)$, which is the space of all continuous maps from the interval $[0,1]$ into X, endowed with compact-open topology. Then there is the evaluation map $ev : X^I \to X \times X$ given $ev(f) = (f(0), f(1))$ for $f \in X^I$ which is a fibration. It induces the pullback diagram, by denoting $LX = \{f \in Map([0,1], X); f(0) = f(1)\}$;

$$\begin{array}{ccc} LX & \longrightarrow & X^I \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle ev} \\ X & \xrightarrow[\Delta]{} & X \times X, \end{array} \tag{3.3}$$

where $p : LX \to X$ is the evaluation at a base pint $* \in X$ Δ is the diagram map,i.e., $\Delta(x) = (x, x)$ for $x \in X$. There is a natural injection $c : X \to X^I$, sending a point $x \in X$ into the constant path at x, which embeds X in X^I as a deformation retract. Thus we have the fiber square;

$$\begin{array}{ccc} LX & \longrightarrow & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \Delta} \\ X & \xrightarrow[\Delta]{} & X \times X. \end{array} \tag{3.4}$$

Associated to this fiber square (3.4),

COROLLARY 3.4. *There is a spectral sequence with $E_2^{p,q} = Tor^{p,q}_{H^*(X \times X;\mathbf{k})}(H^*(X; \mathbf{k}), H^*(X; \mathbf{k})) = Tor^{p,q}_{H^*(X;\mathbf{k}) \otimes H^*(X;\mathbf{k})}(H^*(X; \mathbf{k}), H^*(X; \mathbf{k}))$ which converges to $H^*(LX, ; \mathbf{k})$ as algebras.*

In Sections 3,4 of Chapter 1 and Sections 2,3 of Chapter 2, we denote the Eilenberg-Moore spectral sequence of (3.4) over $\mathbf{k} = \mathbf{Z}/p$ by

$$\{E_r^{*,*}, d_r\}.$$

This spectral sequence is lying in the second quadrant, that is, $E_r^{p,q}$ is bigraded with $p \leq 0$ and $q \geq 0$ and the bidegree of d_r is $(r, 1-r)$. We may call the indexes p and $p+q$ the *filtration degree* and the *total degree*, respectively. There exists a decreasing filtration

$$\{F^iH^*(LX;\mathbf{Z}/p)\}_{i\le 0}$$

on $H^*(LX;\mathbf{Z}/p)$ such that

$$E_0^{i,j} := F^iH^{i+j}(LX;\mathbf{Z}/p)/F^{i+1}H^{i+j}(LX;\mathbf{Z}/p).$$

The limit term $E_\infty^{*,*}$ is isomorphic to $E_0^{*,*}$ as bigraded algebras.

Finally, we remark that if $p=0$ and X is formal (see Section 1), there is a chain of quasi-isomorphisms:

$$H^*(X;\mathbf{Q}) \overset{\psi}{\leftarrow} \mathcal{M}(X) \overset{\phi}{\to} (\Omega^*(X),\partial).$$

It induces a chain of isomorphisms:

$$\begin{aligned} Tor_{H^*(X;\mathbf{Q})\otimes H^*(X;\mathbf{Q})}(H^*(X;\mathbf{Q}),H^*(X;\mathbf{Q})) &\overset{\cong}{\leftarrow}_{(a)} Tor_{\mathcal{M}(X)\otimes\mathcal{M}(X)}(\mathcal{M}(X),\mathcal{M}(X)) \\ &\overset{\cong}{\to}_{(b)} Tor_{\Omega^*(X)\otimes\Omega^*(X)}((\Omega^*(X),\partial),(\Omega^*(X),\partial)) \\ &\cong Tor_{(\Omega^*(X\times X),\partial)}((\Omega^*(X),\partial),(\Omega^*(X),\partial)) \\ &\overset{\cong}{\to}_{(c)} H^*(LX;\mathbf{Q}), \end{aligned}$$

as algebras by the argument of [**44**, Section 2]. Here (a) and (b) are induced by ψ and ϕ, respectively. Also (c) is θ_* in Proposition 3.1 constructed for the square (3.4). We see, in (2.1) of the next chapter, that the left hand is the Hochschild homology algebra $HH_*(H^*(X;\mathbf{Q}),0)$ of the $\mathbf{Q}$-CDGA $(H^*(X;\mathbf{Q}),0)$ with trivial differential.

CHAPTER 1

The mod p cohomology algebras of free loop spaces

1. Notations

Let $\mathbf{k}_p$ be a field of characteristic p and (C,d) a differential graded commutative algebra (DGA) over $\mathbf{k}_p$ endowed with a differential d of degree $+1$.

Let $\Lambda(y_1,...,y_l)=\Lambda_{\mathbf{k}_p}(y_1,...,y_l)$ be the exterior algebra over $\mathbf{k}_p$, where $y_i^2=0$ and $y_iy_j=(-1)^{ab}y_jy_i$ for $\deg y_i=a$ and $\deg y_j=b$. Let $\Gamma[\omega_1,...,\omega_m]=\Gamma_{\mathbf{k}_p}[\omega_1,...,\omega_m]$ be the divided power algebra over $\mathbf{k}_p$. Note that, as a vector space, $\Gamma[\omega]$ is generated by elements $\gamma_i(\omega)$ $(i>0)$ and a unit $\gamma_0(\omega)=1$, and the multiplication is defined by

$$\gamma_k(\omega)\gamma_l(\omega)=\binom{k+l}{k}\gamma_{k+l}(\omega).$$

Furthermore $\Gamma^+[\omega_1,...,\omega_s]$ denotes the subalgebra of $\Gamma[\omega_1,...,\omega_s]$ generated by the monomials $\{\gamma_{k_1}(\omega_1)\cdots\gamma_{k_s}(\omega_s) : k_1>0,..,k_s>0\}$ ([**1**]). When $p=0$, we regard the algebra $\Gamma[\omega_1,...,\omega_m]$ and an element $\gamma_k(\omega_i)$ in $\Gamma[\omega_1,...,\omega_m]$ as the polynomial algebra $\mathbf{k}_0[\omega_1,...,\omega_m]$ and ω_i^k, respectively.

For any algebra B, let A, I and S be a subalgebra, an ideal and a subset of B, respectively. Then A/I denotes the quotient algebra of A by the ideal $A\cap I$ and $(S)_A$ denotes the sub$A$-module of B generated by S when we regard B as an A-module. If $A=B$, $(S)_A$ is the ideal of A generated by S. For an algebra A and elements $a_1,..,a_s$ of A, $Ann_A(a_1,..,a_s)$ denotes the ideal of A generated by the elements $\{a\in A;\ a\cdot a_i=0 \text{ for } 1\le i\le s\}$. A sequence of polynomials $\rho_1,..,\rho_m$ of $\mathbf{k}_p[x_1,...,x_n]$ is said *regular* if

$$Ann_{\mathbf{k}_p[x_1,...,x_n]/(\rho_1)}(\rho_2)=0,\quad Ann_{\mathbf{k}_p[x_1,...,x_n]/(\rho_1,\rho_2)}(\rho_3)=0,$$

$$Ann_{\mathbf{k}_p[x_1,...,x_n]/(\rho_1,\rho_2,\rho_3)}(\rho_4)=0,\cdots,Ann_{\mathbf{k}_p[x_1,...,x_n]/(\rho_1,..,\rho_{m-1})}(\rho_m)=0.$$

Let $\mathbf{Z}$ be the integer ring and p a prime number. A graded commutative algebra over $\mathbf{Z}/p$, A, is said to have a *p-simple system of generators*, $\{x_i; i\in J\}$, if A is generated as a vector space over $\mathbf{Z}/p$ by the monomials $x_{i_1}^{m_1} x_{i_1}^{m_1} \cdots x_{i_1}^{m_1}$ where $i_1 < ... < i_k$ and $1 \leq j \leq k$, $0 \leq m_j \leq p-1$, if deg x_{i_j} is even, and $0 \leq m_j \leq 1$, if deg x_{i_j} is odd. For example, p-simple system of generators of the polynomial algebra $\mathbf{Z}/p\,[u]$ with deg u even is given by $\{1, u, u^p, u^{p^2}, ...\}$.

We recall that the mod 2 cohomology operation $Sq^i : H^*(\ ;\mathbf{Z}/2) \to H^{*+i}(\ ;\mathbf{Z}/2)$, for $i \geq 0$, satisfies the following: (1) If $x \in H^n(X;\mathbf{Z}/2)$, then $Sq^n x = x^2$, (2) If $x \in H^n(X;\mathbf{Z}/2)$ and $i > n$, then $Sq^i x = 0$, (3) for all $x, y \in H^*(X;\mathbf{Z}/2)$, $Sq^k(x \cup y) = \sum_{i=0}^k Sq^i x \cup Sq^{k-i} y$. Also, for an odd prime p, we recall that the mod p cohomology operation $P^i : H^*(\ ;\mathbf{Z}/p) \to H^{*+2i(p-1)}(\ ;\mathbf{Z}/p)$, for $i \geq 0$, especially satisfies the condition; if $x \in H^{2n}(X;\mathbf{Z}/p)$, then $P^n x = x^p$. For the Eilenberg-Moore spectral sequence $(E_r^{p,q}, d_r)$ over $\mathbf{Z}/p$ of (3.1) in Chapter 0, we have the operations keep the filtrations, that is, $Sq^i : E_r^{p,q} \to E_r^{p,q+i}$ and $P^i : E_r^{p,q} \to E_r^{p,q+2i(p-1)}$ ([**40**, Theorem 1.2]). Since the filtration on $H^*(LX;\mathbf{Z}/p)$ is induced as the limit term of the Eilenberg-Moore spectral sequence, we have

$$\begin{aligned} Sq^i &: F^pH^*(LX;\mathbf{Z}/2) \to F^pH^{*+i}(LX;\mathbf{Z}/2) \\ P^i &: F^pH^*(LX;\mathbf{Z}/p) \to F^pH^{*+2i(p-1)}(LX;\mathbf{Z}/p). \end{aligned} \tag{1.1}$$

Finally we recall the definition of *p-formal* (or *$\mathbf{Z}/p$-formal*).

DEFINITION 1.1 ([**2**, Lemma 9]). *A space X is said p-formal (or $\mathbf{Z}/p$-formal) if the singular cochain complex $C^*(X;\mathbf{Z}/p)$ and $(H^*(X;\mathbf{Z}/p), 0)$ are connected by a chain of $\mathbf{Z}/p$-DGA-quasi-isomorphisms.*

The Eilenberg-Moore map θ in Chapter 0 induces an isomorphism of algebra from $Tor_{C^*(X;\mathbf{Z}/p)\otimes C^*(X;\mathbf{Z}/p)}(C^*(X;\mathbf{Z}/p), C^*(X;\mathbf{Z}/p))$ to $H^*(LX;\mathbf{Z}/p)$. Therefore if X is p-formal, we have

$$H^*(LX;\mathbf{Z}/p) \cong Tor_{H^*(X;\mathbf{Z}/p)\otimes H^*(X;\mathbf{Z}/p)}(H^*(X;\mathbf{Z}/p), H^*(X;\mathbf{Z}/p)) \tag{1.2}$$

as a vector space ([**22**, Proposition 3.1]). Consequently, the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ of the square (3.4) of Chapter 0 collapses at the E_2-term.

In this thesis, " formal" means "$\mathbf{Q}$-formal", unless we explicitly mention otherwise.

2. Hochschild homology

Let $C = \oplus_{\geq 0} C^i$ a graded $\mathbf{k}_p$-algebra and $\mathcal{C} = (C, d)$ a $\mathbf{k}_p$-differential graded algebra($\mathbf{k}_p$-DGA) endowed with a differential d of degree 1. Then we denote the Hochschild homology of (C, d) ([**14**], [**5**], [**6**]) by $HH_*(C, d)$, which is defined as follows: First we define the normalized bar complex $(\mathbf{N}(C), b)$ of $\mathcal{C}$:

$$\mathbf{N}(C) = \sum_{k=0}^{\infty} C \otimes \bar{C}^{\otimes k}$$

and the differential $b = b_0 + b_1$ have the properties such that

$$b_0([z_0, .., z_k]) = -\sum_{i=0}^{k} (-1)^{\varepsilon_{i-1}} [z_0, .., z_{i-1}, d(z_i), z_{i+1}, .., z_k] \text{ and}$$

$$b_1([z_0, .., z_k]) = -\sum_{i=0}^{k-1} (-1)^{\varepsilon_i} [z_0, .., z_{i-1}, z_i z_{i+1}, z_{i+2}, .., z_k] + (-1)^{(deg z_k - 1)\varepsilon_{k-1}} [z_k z_0, .., z_{k-1}],$$

where $\bar{C} = \oplus_{i>0} C^i$, $\deg[z_0, .., z_k] = \deg z_0 + \cdots + \deg z_k - k$, for $[z_0, .., z_k]$ in $\mathbf{N}(C)$, and $\varepsilon_i = \deg z_0 + \cdots + \deg z_i - i$. Here $(\mathbf{N}(C), b)$ is bigraded by

$$\text{bideg } [z_0, .., z_k] = (-k, \deg z_0 + ... + \deg z_k)$$

for $[z_0, .., z_k] \in (\mathbf{N}(C), b)$.

Let C^{op} be the opposite algebra of C, where the product of a and b is given by $a \cdot b = (-1)^{\deg a \cdot \deg b} ba$, and $\mathcal{C}^{op} = (C^{op}, d)$. Then $C \otimes C^{op}$-module structure of C is given by $(a \otimes b) \cdot c = (-1)^{\deg c \cdot \deg b} acb$. It is known $HH_*(\mathcal{C}) \cong Tor_{\mathcal{C}\otimes\mathcal{C}^{op}}(\mathcal{C}, \mathcal{C})$ ([**30**, 1.1.13, 5.3.2]). In fact the bi-complex $\{\mathcal{C}^{\otimes n}, b'\}_{n \geq 2}$, where $b' = b_0 + b_1'$ with

$$b_1'([z_0, .., z_k]) = -\sum_{i=0}^{k-1} (-1)^{\varepsilon_i} [z_0, .., z_{i-1}, z_i z_{i+1}, z_{i+2}, .., z_k]$$

is a bar complex of $\mathcal{C}$ as a $\mathcal{C} \otimes \mathcal{C}^{op}$-module ([**30**, 1.1.11]). If C is commutative, we have $HH_*(\mathcal{C}) \cong Tor_{\mathcal{C}\otimes\mathcal{C}}(\mathcal{C}, \mathcal{C})$. Since $H_*(\mathcal{C}) = (H_*(\mathcal{C}), 0)$ is commutative, when $H_*(\mathcal{C})$ is considered as a module over $H_*(\mathcal{C}) \otimes H_*(\mathcal{C})$, we have

$$HH_*(H_*(\mathcal{C})) = HH_*(H_*(\mathcal{C}), 0) \cong Tor_{H_*(\mathcal{C})\otimes H_*(\mathcal{C})}(H_*(\mathcal{C}), H_*(\mathcal{C})) \tag{2.1}$$

as bigraded algebras with appropriate bigradings.

From Corollary 3.4 of Chapter 0 and (2.1), we have the E_2-term of the Eilenberg-Moore spectral sequence for (3.4) in Chapter 0 is given as

$$E_2^{*,*} \cong HH_*(H^*(X;\mathbf{k}_p),0)$$

as bigraded algebras with appropriate bigradings.

If $\mathcal{C}$ is the rational de Rham complex $(\Omega^*(X),\partial)$,

$$\begin{aligned} HH_*(\Omega^*(X),\partial) &\cong Tor_{(\Omega^*(X),\partial)\otimes(\Omega^*(X),\partial)}((\Omega^*(X),\partial),\Omega^*(X),\partial)) \\ &\cong Tor_{(\Omega^*(X\times X),\partial)}((\Omega^*(X),\partial),(\Omega^*(X),\partial)) \\ &\cong H^*(LX;\mathbf{Q}) \end{aligned}$$

as bigraded algebras with appropriate bigradings.

A *graded complete intersection algebra* over $\mathbf{k}_p$ ($\mathbf{k}_p$-GCI-algebra) is a commutative graded algebra:

$$\Lambda = \Lambda(y_1,..,y_l)\otimes\frac{\mathbf{k}_p[x_1,..,x_n]}{(\rho_1,..,\rho_m)}$$

where $\rho_1,..,\rho_m$ is a regular sequence (or $m=0$) and where deg y_j is odd, deg x_i is even if $p\neq 2$ and $l=0$ if $p=2$. We say that Λ is *simply connected* if $\Lambda^1=0$.

If $H^*(X;\mathbf{k}_p)$ is a $\mathbf{k}_p$- GCI-algebra, there is a commutative differential graded algebra over $\mathbf{k}_p$ ($\mathbf{k}_p$-CDGA), $\mathcal{K}$ as follows:
Put

$$A = \mathbf{k}_p[x_1,..,x_n]/(\rho_1,..,\rho_m)\otimes\Lambda(\bar{x}_1,..,\bar{x}_n),$$

where deg $\bar{y}_i$ = deg $y_i - 1$, deg $\bar{x}_i$ = deg $x_i - 1$. Then the Koszul-Tate complex associated to the GCI-algebra Λ is defined as

$$\mathcal{K} = (\Lambda(y_1,..,y_l)\otimes\Gamma[\bar{y}_1,..,\bar{y}_l]\otimes A\otimes\Gamma[\omega_1,..,\omega_m],d),$$

where deg ω_j = deg $\rho_j - 2$, $d(y_i)=d(\bar{y}_i)=d(A)=0$ and $d(\omega_j)=\sum_{i=1}^{n}\frac{\partial\rho_j}{\partial x_i}\bar{x}_i$. Then we have

PROPOSITION 2.1 ([**44**], [**25**]). *$H_*(\mathcal{K})\cong HH_*(\Lambda,0)$ as algebras.*

We remark this is isomorphic also as a bigraded algebra, if the bidegree of $\mathcal{K}$ is given by bideg $x_i=(0,\deg x_i)$, bideg $y_j=(0,\deg y_j)$, bideg $\bar{x}_i=(-1,\deg x_i)$, bideg $\bar{y}_j=(-1,\deg y_j)$, and bideg $\omega_k=(-2,\deg\rho_k)$.

We call $\mathcal{K}$ the *Koszul-Tate complex* of Λ.

3. A useful subalgebra of Hochschild homology

The proof of the following proposition is based upon the projective resolution of Λ as a $\Lambda\otimes\Lambda$ -module, constructed in [**44**] (see also [**25**] if $p\neq 0$).

PROPOSITION 3.1. (i) Suppose $\Lambda=\Lambda(y_1,..,y_l)\otimes\mathbf{k}_p[x_1,..,x_n]/(\rho_1,..,\rho_m)$ is a simply connected GCI-algebra, where ρ_i is decomposable for any i. Then there exists a monomorphism of algebras

$$\begin{aligned} \psi : B = {} & \Lambda(y_1,..,y_l)\otimes\Gamma[\bar{y}_1,..,\bar{y}_l] \\ & \otimes\left\{\left(A\oplus\sum_{s=1}^{m}\sum_{i_1<\cdots<i_s}Ann_A\left(d(\omega_{i_1}),..,d(\omega_{i_s})\right)\otimes\Gamma^+[\omega_{i_1},..,\omega_{i_s}]\right)\Big/(d\Gamma[\omega_1,..,\omega_m])_A\right\} \\ & \hookrightarrow Tor_{\Lambda\otimes\Lambda}(\Lambda,\Lambda)=HH_*(\Lambda,0), \end{aligned}$$

where $A=\mathbf{k}_p[x_1,..,x_n]/(\rho_1,..,\rho_m)\otimes\Lambda(\bar{x}_1,..,\bar{x}_n)$, $d(\omega_j)=\sum_{i=1}^{n}\frac{\partial\rho_j}{\partial x_i}\bar{x}_i$, deg $\bar{y}_i$ = deg y_i-1, deg $\bar{x}_i$ = deg x_i-1 and deg ω_j = deg ρ_j-2.

(ii) In the case $\Lambda=\Lambda(y_1,..,y_l)\otimes\mathbf{k}_p[z_1,..,z_m]\otimes\mathbf{k}_p[x_1,..,x_n]/(x_1^{s_1+1},..,x_n^{s_n+1})$, there exists an isomorphism of algebras

$$\begin{aligned} \psi : B = {} & \Lambda(y_1,..,y_l)\otimes\Gamma[\bar{y}_1,..,\bar{y}_l]\otimes\mathbf{k}_p[z_1,..,z_m]\otimes\Lambda(\bar{z}_1,..,\bar{z}_m)\otimes \\ & \bigotimes_{i=1}^{n}\left\{A_i/((s_i+1)x_i^{s_i}\bar{x}_i)\oplus\left(\left(\epsilon_i,x_i,\bar{x}_i\right)_{A_i}\Big/\left((s_i+1)x_i^{s_i}\bar{x}_i\right)_{A_i}\right)\otimes\Gamma^+[\omega_i]\right\} \\ & \cong Tor_{\Lambda\otimes\Lambda}(\Lambda,\Lambda)=HH_*(\Lambda,0), \end{aligned}$$

where $A_i=\mathbf{k}_p[x_i]/(x_i^{s_i+1})\otimes\Lambda(\bar{x}_i)$, the element ϵ_i is the unit 1 in A_i if $s_i+1=0$ in $\mathbf{k}_p$, otherwise, is zero.

PROOF. (i) From Proposition 2.1, there exists a natural inclusion ψ. Let C be $A \oplus \sum_{s=1}^{m} \sum_{i_1 < \cdots < i_s} Ann_A(d(\omega_{i_1}), .., d(\omega_{i_s})) \otimes \Gamma^+[\omega_{i_1}, .., \omega_{i_s}]$. Then we note that the ideal $(d\Gamma[\omega_1, .., \omega_m])_{A \otimes \Gamma[\omega_1, .., \omega_m]} \cap C$ of C is equal to $(d\Gamma[\omega_1, .., \omega_m])_A \cap C$ in $A \otimes \Gamma[\omega_1, .., \omega_m]$.

(ii) Let Λ be a truncated algebra $\mathbf{k}_p[x_i]/(x_i^{s_i+1})$. By the direct calculation, we see that $HH_*(\Lambda, 0)$ is isomorphic to $A_i/((s_i+1)x_i^{s_i}\bar{x}_i) \oplus (\epsilon_i, x_i, \bar{x}_i)_{A_i}/((s_i+1)x_i^{s_i}\bar{x}_i)_{A_i} \otimes \Gamma^+[\omega_i]$ as an algebra. From the Künneth theorem for the Hochschild homology, we have the isomorphism ψ. □

The following example shows that the monomorphism ψ in Proposition 3.1 (i) is not an isomorphism in general. Consider the commutative graded $\mathbf{k}_0$-algeba $A = \mathbf{k}_0[x, y]/(x^4 + y^2, y^4) \otimes \Lambda(\bar{x}, \bar{y})$ over a field $\mathbf{k}_0$ of characteristic zero, where $\deg x = 2$, $\deg y = 4$, $\deg \bar{x} = 1$ and $\deg \bar{y} = 3$. Let (C, d) be a differential graded algebra $(A \otimes \mathbf{k}_0[\omega_1, \omega_2], d)$ endowed with a differential d of degree $+1$, satisfying

$$d(\omega_1) = (\frac{\partial}{\partial x}\bar{x} + \frac{\partial}{\partial y}\bar{y})(x^4 + y^2) = 4x^3\bar{x} + 2y\bar{y} \quad \text{and}$$

$$d(\omega_2) = (\frac{\partial}{\partial x}\bar{x} + \frac{\partial}{\partial y}\bar{y})y^4 = 4y^3\bar{y},$$

where $\deg \omega_1 = 6$ and $\deg \omega_2 = 14$. The element

$$\alpha = 2y^2\bar{x}\omega_1 - \bar{x}\omega_2$$

is a cycle element with degree 15 in C. In fact,

$$d(\alpha) = -2y^2\bar{x}(4x^3\bar{x} + 2y\bar{y}) + \bar{x}(4y^3\bar{y}) = -4y^3\bar{x}\bar{y} + 4y^3\bar{x}\bar{y} = 0.$$

If there exists an element β such that $d(\beta) = \alpha$, then β must have the elements ω_2^k $(k \geq 2)$ or $\omega_2^k\omega_1^s$ $(s \geq 1, k \geq 1)$ as terms since α has the non-zero term $\bar{x}\omega_2$. Though the degree of β is 14, $\deg \omega_2^k = 14k > 14$ and $\deg \omega_2^k\omega_1^s = 14k + 6s > 14$. Therefore α represents a non-zero element of $H_*(C, d)$. Let

$$\begin{aligned}\Gamma = A &\oplus Ann_A(d\omega_1) \otimes \mathbf{k}_0{}^+[\omega_1] \\ &\oplus Ann_A(d\omega_2) \otimes \mathbf{k}_0{}^+[\omega_2] \\ &\oplus Ann_A(d\omega_1, d\omega_2) \otimes \mathbf{k}_0{}^+[\omega_1, \omega_2].\end{aligned}$$

If the monomorphism ψ is an isomorphism, then there exists an element $\gamma \in \Gamma$ which maps $\alpha + d(\beta)$ by the lifting map of $\psi : \Gamma \to H^*(C, d)$ for some element $\beta \in C$. Since the degree of γ is 15, γ can be written as $b_0 + b_1\omega_1 + b_2\omega_1^2 + b_3\omega_2$, where $b_0 \in A$, $b_1, b_2 \in Ann_A(d(\omega_1))$ and $b_3 \in Ann_A(d(\omega_2))$. Then

$$\begin{aligned}\psi^{-1}d(\beta) &= \gamma - \psi^{-1}(\alpha) \\ &= b_0 + (b_1 - 2y^2\bar{x})\omega_1 + b_2\omega_1^2 + (b_3 + \bar{x})\omega_2.\end{aligned}$$

Applying the above argument about degrees again, we have $b_3 = -\bar{x}$. On the other hand $\bar{x} \notin Ann_A(d(\omega_2)) = Ann_A(4y^3\bar{y})$, which is a contradiction.

REMARK 3.2. *Let $H^*(X; \mathbf{k}_0)$ be isomorphic to a GCI-algebra over a field $\mathbf{k}_0$ $\Lambda = \Lambda(y_1, .., y_l) \otimes \mathbf{k}_0[x_1, .., x_n]/(\rho_1, .., \rho_m)$, where ρ_i is decomposable. Here we denote the $\mathbf{k}_0$-minimal model of $(H^*(X; \mathbf{k}_0), 0)$ by $\mathcal{M} = (\wedge V, d)$. It is given as follows:*

$$\wedge V = \Lambda(y_1, .., y_l) \otimes \mathbf{k}_0[x_1, .., x_n] \otimes \Lambda(\tau_1, .., \tau_m), d(y_i) = d(x_i) = 0 \text{ and } d(\tau_j) = \rho_j.$$

Then the cohomology of LX over $\mathbf{k}_0$, is equal the cohomology of a complex $\varepsilon(\mathcal{M}) = (\wedge V \otimes \wedge \bar{V}, \delta)$ ([7], [49]). This $\mathbf{k}_0$-CDGA has the following properties (compare Section 2 of Chapter 0):

(i) $\bar{V}^i = V^{i+1}$, that is, $\wedge \bar{V} = \mathbf{k}_0[\bar{y}_1, .., \bar{y}_l] \otimes \Lambda(\bar{x}_1, .., \bar{x}_n) \otimes \mathbf{k}_0[\bar{\tau}_1, .., \bar{\tau}_m]$.

(ii) $\delta(y_i) = \delta(x_i) = \delta(\bar{y}_i) = \delta(\bar{x}_i) = 0$, $\delta(\tau_j) = \rho_j$ and $\delta(\bar{\tau}_j) = -\sum_{i=1}^{n} \frac{\partial \rho_j}{\partial x_i}\bar{x}_i$.

(iii) $H_*(\varepsilon(\mathcal{M})) \cong H^*(LX; \mathbf{k}_0)$ as algebras.

Let $\mathcal{K}$ be the Koszul Tate complex of Λ. In this case, we see that there is a natural $\mathbf{k}_0$-CDGA morphism from $\varepsilon(\mathcal{M})$ to $\mathcal{K}$ which is given by the correspondences: $x_i \mapsto x_i$, $y_i \mapsto y_i$, $\tau_j \mapsto 0$, $\bar{x}_i \mapsto \bar{x}_i$, $\bar{y}_i \mapsto \bar{y}_i$, $\bar{\tau}_j \mapsto \omega_j$ and $\delta \mapsto -d$. This map induces $H_*(\varepsilon(\mathcal{M})) \cong H_*(\mathcal{K})$ as algebras. Hence we see that

$$H^*(LX; \mathbf{k}_0) \cong H_*(\varepsilon(\mathcal{M})) \cong H_*(\mathcal{K}) \cong HH_*(H^*(X; \mathbf{k}_0), 0)$$

as alegras from Proposition 2.1.

4. The mod p cohomology of LX

Let V be a vector space and x, y elements of V. In the proofs of Theorems 4.1, 4.3, 4.5 and 4.8, we will say that "x **contains** y" if the element x can be represented by a linear combination in which the element y has a nonzero coefficient.

By virtue of Proposition 3.1 (ii), first we determine the mod p cohomology of a space LX of free loops on a space X whose mod p cohomology ring is generated by a one element, in Theorems 4.1 and 4.3 below.

THEOREM 4.1. *Let X be a simply connected space whose mod p cohomology is isomorphic to $\Lambda(y)$, where $\deg y$ is odd. Then*

$$H^*(LX; \mathbf{Z}/p) \cong \Lambda(y) \otimes \Gamma[\bar{y}]$$

as an algebra, where $\deg \bar{y} = \deg y - 1$.

PROOF. By Proposition 3.1 (ii), we have

$$E_2^{*,*} \cong \Lambda(y) \otimes \Gamma[\bar{y}],$$

where bideg $y = (0, \deg y)$ and bideg $\bar{y} = (-1, \deg y)$. Suppose that $d_r(\gamma_i(\bar{y}))$ contains the element $y^l \gamma_j(\bar{y})$ for $l = 0$ or $l = 1$. Then we have

$$i(\deg y - 1) + 1 = l \deg y + j(\deg y - 1)$$

$$\text{and} \qquad -i + r = -j$$

by the argument on total degrees and filtration degrees respectively. Then we have $1 + r = (r + l)\deg y$. If $l = 0$, we have $r \deg y = 1 + r$. It contradicts the assumption where $r > 1$ and $\deg y > 2$. If $l = 1$, we have $\deg y = 1$. It contradicts the assumption where X is simply connected. Then $d_r = 0$ for any $r \geq 2$. Thus we can conclude that $E_2^{*,*} \cong E_\infty^{*,*} \cong E_0^{*,*}$ as bigraded algebras.

Let us solve extension problems. In this case, it suffices to prove that $\gamma_{p^f}(\bar{y})^p$ does not contain $y^l \gamma_k(\bar{y})$, for $l = 0$ or 1, since E_0 contains $\Gamma[\bar{y}]$ as a subalgebra and the relations of

$\Gamma[\bar{y}]$ as an algebra are $\{\gamma_{p^f}(\bar{y})^p = 0;\ f \geq 0\}$. Suppose that $\gamma_{p^f}(\bar{y})^p$ contains $y^l \gamma_k(\bar{y})$. Then we have an equality of the total degrees:

$$(T) \quad p^{f+1}(\deg y - 1) = l \deg y + k(\deg y - 1).$$

Since the filtration induced on $H^*(LX; \mathbf{Z}/p)$ as the limit term of the Eilenberg-Moore spectral sequence is invariant under the action of the Steenrod operations (1.1), it follows that $\gamma_{p^f}(\bar{y})^p$ is in filtration $F^{-p^f} H^*(LX; \mathbf{Z}/p)$. Thus we have an inequality of the filtration degrees:

$$(F) \quad p^f \geq k.$$

If $l = 0$, we have from (F) and (T)

$$k(\deg y - 1) < p^{f+1}(\deg y - 1) = k(\deg y - 1),$$

which clearly contradicts. If $l = 1$, we have from (T)

$$p^{f+1}(\deg y - 1) = \deg y + k(\deg y - 1) = (k+1)(\deg y - 1) + 1,$$

that is, $(p^{f+1} - (k+1))(\deg y - 1) = 1$, which contradicts the assumption where $\deg y > 2$ since $\deg y$ is odd. □

REMARK 4.2. *We showed $E_2^{*,*} \cong E_\infty^{*,*}$ by degree arguments. In this case, X is $\mathbf{Z}/p$-formal since $\tilde{H}^i(X; \mathbf{Z}/p)$ is clearly zero whenever i is outside an interval of the form $[k+1, 3k+1]$ for $k = \deg y - 1$ (*[**2**, Lemma 9]*). Then the isomorphism $E_2^{*,*} \cong E_\infty^{*,*}$ is guaranteed from (1.2).*

THEOREM 4.3. *Let X be a simply connected space whose mod p cohomology is isomorphic to $\mathbf{Z}/p[x]/(x^{s+1})$.*
(i) When $s + 1 \equiv 0$ mod p and when $p \neq 2$ or $\deg x \neq 2$,

$$H^*(LX; \mathbf{Z}/p) \cong \mathbf{Z}/p[x] \Big/ (x^{s+1}) \ \otimes \Lambda(\bar{x}) \otimes \Gamma[\omega]$$

as an algebra, where $\deg \bar{x} = \deg x - 1$ and $\deg \omega = (s+1)\deg x - 2$.

(ii) When $s+1 \not\equiv 0$ mod p and when $s > 1$ or $\deg x \neq 2$,

$$H^*(LX;\mathbf{Z}/p) \cong \{ \left(\mathbf{Z}/p[x]/(x^{s+1}) \otimes \Lambda(\bar{x})\right) \big/ (x^s\bar{x})_A\} \oplus \{ (x,\bar{x})_A/(x^s\bar{x})_A) \} \otimes \Gamma^+[\omega]$$

as an algebra, where $A = \mathbf{Z}/p[x]/(x^{s+1}) \otimes \Lambda(\bar{x})$, $\deg \bar{x} = \deg x - 1$ and $\deg \omega = (s+1)\deg x - 2$.

PROOF. (i) By Proposition 1.1 (ii), we have

$$E_2^{*,*} \cong \mathbf{Z}/p[x] \big/ (x^{s+1}) \otimes \Lambda(\bar{x}) \otimes \Gamma[\omega],$$

where bideg $x = (0, \deg x)$, bideg $\bar{x} = (-1, \deg x)$ and bideg $\gamma_i(\omega) = (-2i, i(s+1)\deg x)$.

First we prove the theorem under the assumption that $\deg x$ is even. Suppose that $d_r(\gamma_i(\omega))$ contains the element $x^l\bar{x}\gamma_j(\omega)$. Then we have

$$i((s+1)\deg x - 2) + 1 = (l+1)\deg x - 1 + j((s+1)\deg x - 2)$$

$$\text{and} \qquad -2i + r = -2j - 1$$

by the argument on total degrees and filtration degrees respectively. We have $i = j + (r+1)/2 > j+1$ from the latter. This contradicts the former since $s > l$. Thus we can conclude that $E_2^{*,*} \cong E_\infty^{*,*} \cong E_0^{*,*}$ as bigraded algebras.

Let us solve extension problems. In this case, it suffices to prove that $\bar{x} \cdot \bar{x}$ does not contain x and $\gamma_{p^f}(\omega)^p$ does not contain $x^l\gamma_k(\omega)$ since E_0 contains $\Gamma[\omega]$ as a subalgebra and the relations of $\Gamma[\omega]$ as an algebra are $\{\gamma_{p^f}(\omega)^p = 0;\ f \geq 0\}$. If $\bar{x} \cdot \bar{x}$ contains x, we have $\deg x = 2$. Then it contradicts since $p \neq 2$ from the assumption. Next suppose that $\gamma_{p^f}(\omega)^p$ contains $x^l\gamma_k(\omega)$. Then we have an equality of the total degrees:

$$(T) \quad p^{f+1}((s+1)\deg x - 2) = l \deg x + k((s+1)\deg x - 2).$$

Since the filtration induced on $H^*(LX;\mathbf{Z}/p)$ as the limit term of the Eilenberg-Moore spectral sequence is invariant under the action of the Steenrod operations (1.1), it follows that $\gamma_{p^f}(\omega)^p$ is in filtration $F^{-2p^f}H^*(LX;\mathbf{Z}/p)$. Thus we have an inequality of the filtration degrees:

$$(F) \quad p^f \geq k.$$

From (T) and (F), we have $p = 2$ and $\deg x = 2$. For there is

$$p^{f+1}((s+1)\deg x - 2) \geq_{(a)} (k+1)((s+1)\deg x - 2)$$

$$= (s+1)\deg x - 2 + k((s+1)\deg x - 2) \geq_{(b)} l \deg x + k((s+1)\deg x - 2)$$

in general. Here (a) is induced from (F) and (b) is induced from $s+1 > l$. Then (a) and (b) are equalities by (T). The inequality of (F) and the equality of (a) induce that $p=2$, $f=0$ and $k=1$. The equality of (b) induces that $s = l$ and $\deg x = 2$. Thus the assumption of Theorem 4.3 (i) implies that $\gamma_{p^f}(\omega)^p$ does not contain $x^l\gamma_k(\omega)$, that is, $\gamma_{p^f}(\omega)^p = 0$ in $H^*(LX;\mathbf{Z}/p)$.

Second we prove the theorem under the assumption that $\deg x$ is odd and $p = 2$. The fact that $d_r(\gamma_i(\omega))$ does not contain $x^l\bar{x}\gamma_j(\omega)$ follows by the same argument as above. Suppose that $d_r(\gamma_i(\omega))$ contains $x^l\gamma_j(\omega)$. Then we have

$$i((s+1)\deg x - 2) + 1 = l \deg x + j((s+1)\deg x - 2)$$

$$\text{and} \qquad -2i + r = -2j$$

by the argument on total degrees and column degrees respectively. We have $i = j + r/2 \geq j+1$ from the latter. This contradicts the former from $s+1 > l$. Thus we can conclude that $E_2^{*,*} \cong E_\infty^{*,*} \cong E_0^{*,*}$ as bigraded algebras.

Let us solve extension problems. In this case, it suffices to prove that $\gamma_{p^f}(\omega)^p$ does not contain either $x^l\gamma_k(\omega)$ or $x^l\bar{x}\gamma_k(\omega)$. The fact that $\gamma_{p^f}(\omega)^p$ does not contain $x^l\gamma_k(\omega)$ follows from the same argument as above. Suppose that $\gamma_{p^f}(\omega)^p$ contains $x^l\bar{x}\gamma_k(\omega)$. Then we have

$$p^{f+1}((s+1)\deg x - 2) = (l+1)\deg x - 1 + k((s+1)\deg x - 2)$$

$$\text{and} \qquad 2p^f > 2k + 1$$

by the argument of total degrees and filtration degrees as above respectively. Then these contradict each other from $s > l$. Thus we can conclude that $\gamma_{p^f}(\omega)^p$ does not contain $x^l\bar{x}\gamma_k(\omega)$, that is, $\gamma_{p^f}(\omega)^p = 0$ in $H^*(LX;\mathbf{Z}/p)$. Thus we have Theorem 4.3 (i).

(ii) By Proposition 3.1 (ii), we have that

$$E_2^{*,*} \cong A \big/ \left((s+1)x^s\bar{x}\right)_A \oplus \{(x,\bar{x})_A \big/ \left((s+1)x^s\bar{x}\right)_A\} \otimes \Gamma^+[\omega],$$

where $A = \mathbf{Z}/p[x]/(x^{s+1}) \otimes \Lambda(\bar{x})$, as a bigraded algebra. Let

$$\begin{aligned} A_{l,i} &= \deg x^l\gamma_i(\omega) = l\deg x + i((s+1)\deg x - 2), \\ B_j &= \deg \bar{x}\gamma_j(\omega) = \deg x - 1 + j((s+1)\deg x - 2) \quad \text{and} \\ C_{m,k} &= \deg \bar{x}x^m\gamma_k(\omega) = (m+1)\deg x - 1 + k((s+1)\deg x - 2). \end{aligned}$$

Then we can conclude that the Eilenberg-Moore spectral sequence $\{E_r^{*,*}, d_r\}$ collapses at the $E_2^{*,*}$-term since the following inequalities hold:

$$A_{l,i} + 1 > B_j \quad (i > j+1),$$

$$B_j + 1 > A_{l,i} \quad (j > i),$$

$$A_{l,i} + 1 > C_{m,k} \quad (i > k+1),$$

$$C_{m,k} + 1 > A_{l,i} \quad (k > i),$$

$$B_j + 1 > C_{m,k} \quad (j > k),$$

$$C_{m,k} + 1 > B_j \quad (k > j),$$

$$A_{l,i} + 1 > A_{k,j} \quad (i > j),$$

$$B_j + 1 > B_i \quad (j > i),$$

$$C_{m,k} + 1 > C_{n,l} \quad (k > l).$$

Here the inequalities in ( ) are induced by the argument of column degrees. Note that last five inequalities have meanings in the only case that $p = 2$ and $\deg x$ is odd. Thus we can conclude that $E_2^{*,*} \cong E_\infty^{*,*} \cong E_0^{*,*}$ as a bigraded algebra.

Let us consider extension problems. We must verify that the following equalities hold in $H^*(LX; \mathbf{Z}/p)$:

(1) $x^s \cdot x\gamma_i(\omega) = 0$,

(2) $x^s \cdot \bar{x}\gamma_i(\omega) = 0$,

(3) $\bar{x} \cdot \bar{x}\gamma_i(\omega) = 0$,

(4) $\bar{x}\gamma_j(\omega) \cdot \bar{x}\gamma_k(\omega) = 0$,

(5) $\bar{x}\gamma_j(\omega) \cdot x^l\gamma_k(\omega) = 0$ if $\binom{j+k}{j} \equiv 0 \bmod p$,

(6) $x^l\gamma_j(\omega) \cdot x^m\gamma_k(\omega) = 0$ if $\binom{j+k}{j} \equiv 0 \bmod p$,

where $i, j, k, l, m > 0$ for (1),(4),(5),(6) and $i \geq 0$ for (2),(3).

Let us verify that equality (1) holds. It suffices to prove that $x^s \cdot x\gamma_i(\omega)$ does not contain either $x^m\gamma_j(\omega)$ or $x^m\bar{x}\gamma_j(\omega)$. Suppose that $x^s \cdot x\gamma_i(\omega)$ contains $x^m\gamma_j(\omega)$. Then we have

$$(s+1)\deg x + i((s+1)\deg x - 2) = m\deg x + j((s+1)\deg x - 2)$$

$$\text{and} \quad i > j$$

by the argument of total degrees and filtration degrees respectively. These contradict each other since $s+1 > m$.

Suppose that $x^s \cdot x\gamma_i(\omega)$ contains $x^m\bar{x}\gamma_j(\omega)$, where $p = 2$ and deg$x$ is odd. Then we have

$$(s+1)\deg x + i((s+1)\deg x - 2) = (m+1)\deg x - 1 + j((s+1)\deg x - 2)$$

$$\text{and} \quad 2i > 2j + 1$$

by the argument of total degrees and filtration degrees respectively. These contradict each other since $s > m$. Thus the equality (1) holds. Applying the same argument as above, it follows that equalities (2), (5) and (6) hold.

Let us verify that equality (3) holds. It suffices to prove that $\bar{x} \cdot \bar{x}\gamma_i(\omega)$ does not contain either $x^l\gamma_j(\omega)$ or $x^l\bar{x}\gamma_j(\omega)$.

Suppose that $\bar{x} \cdot \bar{x}\gamma_i(\omega)$ contains $x^l\gamma_j(\omega)$. Then we have

$$2(\deg x - 1) + i((s+1)\deg x - 2) = l\deg x + j((s+1)\deg x - 2)$$

$$\text{and} \quad i + 1 > j$$

by the argument of total degrees and filtration degrees respectively. Then since $s+1 > l$, we have $i = j$, $l = 1$ and $\deg x = 2$. In this case, it turns out that $\bar{x} \cdot \bar{x}\gamma_i(\omega) = \lambda x\gamma_i(\omega)$ for some constant λ. If $\bar{x} \cdot \bar{x} = 0$ in $H^*(LX; \mathbf{Z}/p)$, then $\lambda\bar{x} \cdot x\gamma_i(\omega) = \bar{x} \cdot (\bar{x} \cdot \bar{x}\gamma_i(\omega)) = (\bar{x} \cdot \bar{x}) \cdot \bar{x}\gamma_i(\omega) = 0$. Since $s > 1$, it follows that $\bar{x} \cdot x\gamma_i(\omega) \neq 0$ in $E_0^{*,*}H^*(LX; \mathbf{Z}/p)$, and therefore in $H^*(LX; \mathbf{Z}/p)$ as well. Hence we have $\lambda = 0$. Thus it suffices to show that $\bar{x} \cdot \bar{x} = 0$ in $H^*(LX; \mathbf{Z}/p)$.

When $p \neq 2$, it is clear that $\bar{x} \cdot \bar{x} = 0$. If $p = 2$ and $\bar{x} \cdot \bar{x} \neq 0$, by the usual argument on total degrees and degrees of filtrations, we see that $\bar{x} \cdot \bar{x} = \mu x$ for some non-zero constant μ. The indecomposable element x in $H^*(LX; \mathbf{Z}/p)$ is the image of the indecomposable element x in $H^*(X; \mathbf{Z}/p)$ by the map π^* induced from the projection of the fibration $\pi : LX \to X$. Let s^* be the homomorphism which is induced from a section $s : X \to LX$ defined by $s(a)(t) = a$

($a \in X$ and $t \in S^1$). Since $\mu x = \mu s^*\pi^*(x) = \mu s^*(x) = s^*(\bar{x}) \cdot s^*(\bar{x})$ in $H^*(X;\mathbf{Z}/p)$, it follows that the element x in $H^*(X;\mathbf{Z}/p)$ is decomposable, which is a contradiction.

Suppose that $\bar{x} \cdot \bar{x}\gamma_i(\omega)$ contains $x^l\bar{x}\gamma_j(\omega)$, where $p = 2$ and $\deg x$ is odd. Then we have

$$2(\deg x - 1) + i((s+1)\deg x - 2) = (l+1)\deg x - 1 + j((s+1)\deg x - 2)$$

$$\text{and} \quad i + 1 > j$$

by the argument of total degrees and filtration degrees respectively. If $i = j$, then from the equality of the total degrees we have $(l-1)\deg x = -1$, which is a contradiction. If $i > j$, then from $s > l$ we have

$$(i-j)((s+1)\deg x - 2) \geq (s+1)\deg x - 2 > l \deg x > (l-1)\deg x + 1,$$

which contradicts the equation of the total degrees.

Thus equality (3) holds. Applying the same argument as above, we see that the equality (4) holds. Thus we have Theorem 4.3 (ii). □

REMARK 4.4. *In the case that $s+1 \equiv 0$ mod p, $p = 2$ and $\deg x = 2$ or $s+1 \not\equiv 0$ mod p, $s = 1$ and $\deg x = 2$, we can see that the Eilenberg-Moore spectral sequence converging to $H^*(LX;\mathbf{Z}/p)$ collapses at the E_2-term. However, we cannot solve extension problems by using the usual argument on total degrees and column degrees of the associated bigraded algebra $E_0^{*,*}$. For example, there is no immediate contradiction to the existence of the relation $\omega^2 = x^s\omega$ when $s+1 \equiv 0$ mod p, $p = 2$ and $\deg x = 2$ or the relations $\bar{x} \cdot \bar{x}\gamma_i(\omega) = x\gamma_i(\omega)$ $(i > 0)$ when $s+1 \not\equiv 0$ mod p, $s = 1$ and $\deg x = 2$.*

Next we consider the algebra structure of $H^*(LX;\mathbf{Z}/p)$ in the case that mod p cohomology of simply connected space X is an exterior algebra generated by two elements x_t and x_u with $t \leq u$ in Theorems 4.5 and 4.8 below. If $\tilde{H}^i(X;\mathbf{Z}/p)$ is zero whenever i is outside an interval of the form $[k+1, 3k+1]$, that is, $t \leq u \leq 2t-2$, then X is p-formal [**2**, Lemma 9]. Therefore the Eilenberg - Moore spectral sequence $\{E_r, d_r\}$ collapses at the E_2-term from (1.2). By solving the extension problem of the Eilenberg -Moore spectral sequence, we have

THEOREM 4.5. *Suppose that the mod p cohomology of a simply connected space X is isomorphic to the exterior algebra $\Lambda(x_t, x_u)$, where $\deg x_t = t$ and $\deg x_u = u$ with $t \leq u \leq 2t-2$. If $p > 3$ or $u \neq 3$, $u \neq 2t-3$ and $p = 3$, then*

$$H^*(LX;\mathbf{Z}/p) \cong \Lambda(x_t, x_u) \otimes \Gamma[\bar{x}_t, \bar{x}_u]$$

as an algebra.

REMARK 4.6. *Under the condition that $t \leq u \leq 2t-2$, the p-formality of X enables us to conclude that the spectral sequence $\{E_r, d_r\}$ collapses at the E_2-term. Note that it is not easy to deduce the above fact under the conditions $t \leq u \leq 2t-2$ from degree consideration as the proof of Theorem 4.1 or 4.3. In fact, in the case $p = 3$, $t = 5$ and $u = 7$, simple degree considerations do not suffice to eliminate the possibility that $d_2(\gamma_3(\bar{x}_5)) = x_7\bar{x}_7 + ...$ in the E_2-term.*

PROOF OF THEOREM 4.5. It suffices to prove that the elements $\gamma_{p^f}(\bar{x}_t)^p$ and $\gamma_{p^f}(\bar{x}_u)^p$ do not contain the element $x_t x_u \gamma_i(\bar{x}_t)\gamma_j(\bar{x}_u)$, where $p^f > i + j \geq 0$ and $f \geq 0$, as in the proof of Theorem 4.3 (i).

If $f = 0$, then $i + j = 0$. If $p > 3$, we have

$$\deg x_t x_u < \deg \bar{x}_t^p \leq \deg \bar{x}_u^p.$$

Therefore we can conclude that $\bar{x}_u^p$ and $\bar{x}_t^p$ do not contain the element $x_t x_u$ if $p > 3$. If $p = 3$, we have that $\deg \bar{x}_u^p = \deg x_t x_u$ if and only if $t = u = 3$ and $\deg \bar{x}_t^p = \deg x_t x_u$ if and only if $u = 2t-3$. So if $u \neq 3$ and $u \neq 2t-3$, we can conclude that $\bar{x}_u^p$ and $\bar{x}_t^p$ do not contain the element $x_t x_u$ since $t \leq u$.

If $f > 0$, since $p \neq 2$ and $t \leq u \leq 2t-2$, we have

$$\deg x_t x_u \gamma_i(\bar{x}_t)\gamma_j(\bar{x}_u) < \deg \gamma_{p^f}(\bar{x}_t)^p \leq \deg \gamma_{p^f}(\bar{x}_u)^p.$$

Therefore $\gamma_{p^f}(\bar{x}_t)^p$ and $\gamma_{p^f}(\bar{x}_u)^p$ do not contain $x_t x_u \gamma_i(\bar{x}_t)\gamma_j(\bar{x}_u)$. It turns out that $\gamma_{p^f}(\bar{x}_t)^p = 0 = \gamma_{p^f}(\bar{x}_u)^p$ in $H^*(LX;\mathbf{Z}/p)$. Thus we have Theorem 4.5. □

REMARK 4.7. *In the case $p = 2$, the Eilenberg - Moore spectral sequence $\{E_r^{*,*}, d_r\}$ collapses at the E_2-term because X is p-formal. However, for instance, we can not decide* *whether $\gamma_2(\bar{x}_4)^2$ is equal to $x_3x_4\bar{x}_3\bar{x}_4$ for $p=2,\ t=3, u=4$ by the usual consideration of degrees.*

Suppose that $H^*(X;\mathbf{Z}/2)$ is isomorphic to the truncated polynomial algebra $\mathbf{Z}/2[x_1,..,x_n]/(x_1^{2^{u_1}},..,x_n^{2^{u_n}})$. Then [**46**, Theorem] asserts that the Eilenberg-Moore spectral sequence collapses at the E_2-term if $Sq^1\equiv 0$ on $H^*(X;\mathbf{Z}/2)$. Moreover, from the argument of the proof, we see that the same conclusion holds if the vector space $(\mathrm{Im}\ Sq^1)^{2^{k+1}m_i+2}$ is zero for any $k\geq 0$ and $1\leq i\leq n$, where $m_i=2^{u_i-1}i-1$. In consequence, we have

THEOREM 4.8. *Suppose that the mod 2 cohomology of a simply connected space X is isomorphic to the exterior algebra $\Lambda(x_t,x_{2t-1})$ where $\deg x_t=t$ and $\deg x_{2t-1}=2t-1$.*
(i) If $Sq^{t-1}x_t=0$ and $t>3$, then as an algebra,

$$H^*(LX;\mathbf{Z}/2)\cong\Lambda(x_t,x_{2t-1})\otimes\Gamma[\bar{x}_t,\bar{x}_{2t-1}].$$

(ii) If $Sq^{t-1}x_t=x_{2t-1}$ and $t>3$ or $Sq^1x_2=x_3$, $Sq^2x_3=0$ and $t=2$, then as an algebra,

$$H^*(LX;\mathbf{Z}/2)\cong\Lambda(x_t,x_{2t-1})\otimes\bigotimes_{i\geq 0}\mathbf{Z}/2[\gamma_{2^i}(\bar{x}_t)]\Big/\left(\gamma_{2^i}(\bar{x}_t)^4\right).$$

Let us compare two different resolutions of the GCI-algebra $\Lambda=\bigotimes_k\Lambda(x_k)$, an exterior algebra over $\mathbf{Z}/2$, before proving Theorem 4.8. Let $B^*(\Lambda\otimes\Lambda,\Lambda)$ denote the bar resolution of Λ ([**34**, p.230]), considered as a left $\Lambda\otimes\Lambda$-module. Let

$$\mathcal{F}=\left(\Lambda\otimes\Lambda\otimes\bigotimes_k\Gamma[\bar{x}_k],d\right)$$

where

$$d(\gamma_i(\bar{x}_k))=(x_k\otimes 1-1\otimes x_k)\gamma_{i-1}(\bar{x}_k).$$

Then $\mathcal{F}\xrightarrow{\mu}\Lambda\to 0$ is a proper projective resolution of Λ, considered as a left $\Lambda\otimes\Lambda$-module, where μ denotes the multiplication on Λ ([**25**, Lemma 1.5]).

LEMMA 4.9. *There exists a morphism of resolutions from $B^*(\Lambda\otimes\Lambda,\Lambda)$ to $\mathcal{F}$, inducing an automorphism ϕ of $Tor_{\Lambda\otimes\Lambda}(\Lambda,\Lambda)$ such that*

$$\phi\left(\overbrace{[x_k\otimes 1-1\otimes x_k|\cdots|x_k\otimes 1-1\otimes x_k]}^{\text{i times}}\right)=\gamma_i(\bar{x}_k).$$

PROOF. Since the elements $z=1\otimes 1[x_k\otimes 1-1\otimes x_k|\cdots|x_k\otimes 1-1\otimes x_k]$ are part of a $\Lambda\otimes\Lambda$-basis of the bar resolution $B^*(\Lambda\otimes\Lambda,\Lambda)$, we can define a morphism ψ from $B^*(\Lambda\otimes\Lambda,\Lambda)$ to $\mathcal{F}$ so that $\psi(z)=1\otimes 1\otimes\gamma_i(\bar{x}_k)$. The morphism ψ induces the required isomorphism ϕ. □

PROOF OF THEOREM 4.8. Using Proposition 3.1, we can determine the algebra structure of $E_2^{*,*}$ explicitly. If $t>3$, then the spectral sequence $\{E_r,d_r\}$ collapses at E_2-term by [**46**, Theorem] since a degree argument shows that $Sq^1=0$.

To solve the extension problem, we use the Steenrod operations $\{Sq^i_{EM}\}_{i\geq 0}$ on the Eilenberg - Moore spectral sequence ([**40**],[**43**]), which are induced from operations on the bar construction. Notice that the operations $Sq^i_{EM}(i\geq 0)$ on $E_\infty^{*,*}$ coincide with the operations on $E_0^{*,*}H^*(LX;\mathbf{Z}/2)$ induced from the ordinary Steenrod operations on $H^*(LX;\mathbf{Z}/2)$. Since

$$Sq_{EM}^{2^f(t-1)}\overbrace{[x_t\otimes 1-1\otimes x_t|\cdots|x_t\otimes 1-1\otimes x_t]}^{2^f\text{ times}}=$$
$$\overbrace{[Sq^{t-1}x_t\otimes 1-1\otimes Sq^{t-1}x_t|\cdots|Sq^{t-1}x_t\otimes 1-1\otimes Sq^{t-1}x_t]}^{2^f\text{ times}}$$

in $E_\infty^{*,*}$, it follows from Lemma 4.9 that $\gamma_{2^f}(\bar{x}_t)^2=0$ if $Sq^{t-1}=0$ and $\gamma_{2^f}(\bar{x}_t)^2=Sq^{2^f(t-1)}\gamma_{2^f}(\bar{x}_t)=\gamma_{2^f}(\bar{x}_{2t-1})$ if $Sq^{t-1}\neq 0$ in $E_0^{*,*}$. Since $Sq^{2t-2}x_{2t-1}=0$ for $t>3$, by the same argument as the above, we see that $\gamma_{2^f}(\bar{x}_{2t-1})^2=0$ in $E_0^{*,*}$.

In order to complete the proof of Theorem 4.8 (i), we must show that $\gamma_{2^f}(\bar{x}_t)^2=0$ and $\gamma_{2^f}(\bar{x}_{2t-1})^2=0$ in $H^*(LX;\mathbf{Z}/2)$ if $Sq^{t-1}=0$. To this end, we verify that $\gamma_{2^f}(\bar{x}_t)^2$ and $\gamma_{2^f}(\bar{x}_{2t-1})^2$ do not contain either $\gamma_i(\bar{x}_t)\gamma_j(\bar{x}_{2t-1})$, $x_t\gamma_i(\bar{x}_t)\gamma_j(\bar{x}_{2t-1})$, $x_{2t-1}\gamma_i(\bar{x}_t)\gamma_j(\bar{x}_{2t-1})$ or $x_tx_{2t-1}\gamma_i(\bar{x}_t)\gamma_j(\bar{x}_{2t-1})$, where $2^f>i+j$.

Suppose $\gamma_{2^f}(\bar{x}_t)^2$ contains $\gamma_i(\bar{x}_t)\gamma_j(\bar{x}_{2t-1})$. Then we have $2^{f+1}(t-1)=i(t-1)+j(2t-2)$ by the argument of total degrees. This contradicts $2^f>i+j$. Suppose $\gamma_{2^f}(\bar{x}_t)^2$ contains

$x_t x_{2t-1}\gamma_i(\bar{x}_t)\gamma_j(\bar{x}_{2t-1})$. Then we have

$$2^{f+1}(t-1) = t + 2t - 1 + i(t-1) + j(2t-2)$$

by the argument on total degrees. Though $(t-1)$ divides the left-hand side of the equation, it does not divide the right-hand side, since $t > 3$. Thus we deduce that $\gamma_{2^f}(\bar{x}_t)^2$ does not contain $x_t x_{2t-1}\gamma_i(\bar{x}_t)\gamma_j(\bar{x}_{2t-1})$. By similar arguments, we can eliminate the other possibilities. Therefore we have $\gamma_{2^f}(\bar{x}_t)^2 = 0$ in $H^*(LX;\mathbf{Z}/2)$ if $Sq^{t-1} = 0$. The usual argument on total degrees and filtration degrees allows us to conclude that $\gamma_{2^f}(\bar{x}_{2t-1})^2 = 0$ in $H^*(LX;\mathbf{Z}/2)$.

To prove Theorem 4.8 (ii), we consider the case where $t = 2$. Though the action of Sq^1 on $H^*(X;\mathbf{Z}/2)$ is not trivial, the vector space $(\text{Im } Sq^1)^{2^{k+1}m_i+2} = 0$ for any $k \geq 0$ and $1 \leq i \leq 2$ because $(\text{Im } Sq^1)^{even} = 0$. Therefore the Eilenberg-Moore spectral sequence collapses at the E_2-term.

Furthermore we can see $\gamma_{2^f}(\bar{x}_3) = \gamma_{2^f}(\bar{x}_2)^2 + P$ for any $f \geq 0$, where P is a polynomial generated by elements x_2, x_3, $\gamma_{2^{f-1}}(\bar{x}_2)$, $\gamma_{2^{f-2}}(\bar{x}_2)$, ..., $\gamma_2(\bar{x}_2)$ and $\bar{x}_2$. From the usual argument on total degrees and filtration degrees it follows that $\gamma_{2^f}(\bar{x}_3)^2 = 0$ in $H^*(LX;\mathbf{Z}/2)$. Thus we can construct an isomorphism η of algebras to $H^*(LX;\mathbf{Z}/2)$ from $H^*(LX;\mathbf{Z}/2) \cong \Lambda(x_2,x_3)\otimes\bigotimes_{i\geq 0}\mathbf{Z}/2[\gamma_{2^i}(\bar{x}_2)]/(\gamma_{2^i}(\bar{x}_2)^4)$ satisfying that $\eta(\gamma_{2^f}(\bar{x}_2)) = \gamma_{2^f}(\bar{x}_2)$ and $\eta^{-1}(\gamma_{2^f}(\bar{x}_3)) = \gamma_{2^f}(\bar{x}_2)^2 + P$. The same argument works in case where $t > 3$. This completes the proof. □

REMARK 4.10. *In the case $t = 2$ or $t = 3$, there are some extension problems which cannot be solved by a mere argument with the Steenrod operation on the Eilenberg-Moore spectral sequence and degree considerations as in the proof of Theorem 4.8. For example, there is the problem of whether $\gamma_2(\bar{x}_2)^2 = \bar{x}_2 x_3$ in the case $t = 2$ or $\gamma_{2^2}(\bar{x}_3)^2 = x_3 x_5 \gamma_2(\bar{x}_5)$ in the case $t = 3$.*

REMARK 4.11. *In this chapter, we are interested in the algebra structures of the cohomologies of free loop spaces. In* [**22**], *it is showed that the sequence of Betti numbers $\{dim_{\mathbf{k}} H_n(LX;\mathbf{k})\}_{n>0}$ is unbounded if $\tilde{H}^i(X;\mathbf{k})$ is not zero whenever i is outside an interval of the form $[k+1, 3k+1]$ for some k. But Theorem 4.8 means that the sequence of Betti numbers of $H^*(LX;\mathbf{Z}/2)$ is unbounded even though $\tilde{H}^i(X;\mathbf{Z}/2)$ is not zero whenever i is outside an interval of the form $[k+1, 3k+1]$. In fact, we can easily see that*

$$
\begin{aligned}
dim_{\mathbf{Z}/2}(\Gamma[\bar{u}_t,\bar{u}_{2t-1}])^i = {} & 1 \quad for \quad i = 0, t-1 \\
& 2 \quad for \quad i = 2t-2, 3t-3 \\
& 3 \quad for \quad i = 4t-4, 5t-5 \\
& 4 \quad for \quad i = 6t-6, 7t-7 \\
& 5 \quad for \quad i = 8t-8, 9t-9 \\
& \vdots \\
& m \quad for \quad i = (2m-2)t-(2m-2), (2m-1)t-(2m-1)
\end{aligned}
$$

for $m > 0$. Since $\Gamma[\bar{u}_t,\bar{u}_{2t-1}] \subset E_2^{*,*} \cong E_\infty^{*,*} \cong H^*(LX;\mathbf{Z}/2)$ as vector spaces, we have

$$\lim_{i\to\infty} \dim_{\mathbf{Z}/2} H^i(LX;\mathbf{Z}/2) = \infty$$

from $t > 1$. Thus, for any space X satisfying the assumptions of Theorems 4.5 and 4.8, we see that the sequence of Betti numbers of $H^*(LX;\mathbf{Z}/p)$ is unbounded for any p in the assumption. If this space X is a closed Riemannian manifold, then for any Riemannian metric there exist infinitely many geometrically distinct closed geodesics on X ([**18**]*).*

5. Hodge decomposition

Let X be a simply connected space and let φ_n the power map $\varphi_n : LX \to LX$ defined by $\varphi_n(\gamma)(e^{i\theta}) = \gamma(e^{in\theta})$. Then we can put $H^*(LX;\mathbf{Q}) = \oplus_{i\geq 0} HH_*^{(i)}$, where $HH_*^{(i)}$ is the eigenspace of the eigenvalue n^i of the power operation φ_n^* ([**5**]). Here $HH_*^{(i)}$ is called *the i-factor of the Hodge decomposition* of the rational cohomology of LX. In general, for the minimal model $\mathcal{M} = (\wedge V, d)$ of X there is a minimal model $\varepsilon(\mathcal{M}) = (\wedge V \otimes \wedge\bar{V}, \delta)$, where $H_*(\varepsilon(\mathcal{M})) \cong H^*(LX;\mathbf{Q})$ (see Remark 1.2). Here $\bar{V}^i = V^{i+1}$. Then we can decompose $\wedge V \otimes \wedge\bar{V}$ as $\oplus_i(\wedge V \otimes \wedge^i\bar{V})$. Since $\delta(\wedge V \otimes \wedge^i\bar{V}) \subset \wedge V \otimes \wedge^i\bar{V}$, we can put $H_*(\wedge V \otimes \wedge\bar{V}, \delta) = \oplus_i H_*(\wedge V \otimes \wedge^i\bar{V}, \delta)$ ([**7**]). Then it is known that

THEOREM 5.1 ([**5**]). *$HH_*^{(i)} \cong H_*(\wedge V \otimes \wedge^i\bar{V}, \delta)$, especially, $HH_*^{(0)} \cong H_*(\wedge V, d) = H^*(X;\mathbf{Q})$.*

We will take advantage of this identification throughout the remainder of this section. We consider only the case in which $H^*(X;\mathbf{Q})$ is a GCI-algebra which is isomorphic to $\Lambda = \mathbf{Q}[x_1,..,x_n]/(\rho_1,..,\rho_m)$. Then $\mathcal{M}$ is uniquely determined by $H^*(X;\mathbf{Q})$, since $H^*(X;\mathbf{Q})$ is then intrinsically formal ([**21**]). This is isomorphic to $\mathcal{M}$ of Remark 3.2 with $\mathbf{k}_0 = \mathbf{Q}$ and $l = 0$.

In the proofs of the following theorems, we use the notation of Proposition 3.1 (i), in particular the correspondence of $\varepsilon(\mathcal{M})$ and $\mathcal{K}$, as made explicit in Remark 3.2.

THEOREM 5.2. *Let $H^*(X;\mathbf{Q})$ be a GCI-algebra $\Lambda = \mathbf{Q}[x_1,..,x_n]/(\rho_1,..,\rho_m)$ where ρ_i is decomposable and let $\overline{HH}_*^{(i)}$ denote the vector space $HH_*^{(i)}/HH_*^{(0)}\cdot HH_*^{(i)}$. Then*

(i) *For $m \le n$*

$$\dim_{\mathbf{Q}} \overline{HH}_*^{(i)} \ge \binom{m-n+i-1}{i-n} \quad \text{when } i > n,$$
$$\dim_{\mathbf{Q}} \overline{HH}_*^{(i)} \ge \binom{n}{i} \quad \text{when } i \le n.$$

In particular, $dim_{\mathbf{Q}} HH_*^{(i)} \neq 0$ for any $i \ge 0$.

(ii) *If $m = n$ then*

$$\dim_{\mathbf{Q}} HH_*^{(i)} \ge \binom{i-1}{i-n} + \binom{n+i-1}{i} \quad \text{when } i > n,$$
$$\dim_{\mathbf{Q}} HH_*^{(i)} \ge \binom{n}{i} + \binom{n+i-1}{i} \quad \text{when } 1 \le i \le n.$$

PROOF. (i) Since $\bar{x}_1\cdots\bar{x}_n$ belongs to $Ann_A(d(\omega_{i_1}),...,d(\omega_{i_s}))$ for any $i_1,..,i_s$, it follows that the elements $\bar{x}_1\cdots\bar{x}_n\omega_1^{k_1}\cdots\omega_m^{k_m}$ $(k_1 \ge 0,..,k_m \ge 0)$ represent elements of $\overline{HH}_*^{(i)}$ where $k_1+\cdots+k_m+n = i$ from Proposition 3.1 (i). Moreover $(d\,\mathbf{Q}[\omega_1,..,\omega_m])_A$ does not contain any linear combination of elements $\bar{x}_1\cdots\bar{x}_n\omega_1^{k_1}\cdots\omega_m^{k_m}$. Therefore Proposition 3.1 (i) also enables us to conclude that the elements $\bar{x}_1\cdots\bar{x}_n\omega_1^{k_1}\cdots\omega_m^{k_m}$ $(k_1 \ge 0,..,k_m \ge 0)$ are linearly independent in $\overline{HH}_*^{(i)}$. Thus $\dim_{\mathbf{Q}} \overline{HH}_*^{(i)} \ge \binom{m-1+i-n}{i-n}$ when $i > n$. Furthermore since the elements $\bar{x}_{j_1}\cdot\bar{x}_{j_2}\cdots\bar{x}_{j_i}$ $(1 \le j_1 < \ldots < j_i \le n)$ are linearly independent in $\overline{HH}_*$, it follows that $\dim_{\mathbf{Q}} \overline{HH}_*^{(i)} \ge \binom{n}{i}$ when $i \le n$.

(ii) Let $[\Lambda]$ be the fundamental class of the GCI-algebra Λ. Since $[\Lambda]$ annihilates the augmentation ideal $\bar{\Lambda}$, it follows from Proposition 3.1 (i) that the elements $[\Lambda]\omega_1^{k_1}\cdots\omega_n^{k_n}$ represent nonzero elements in $HH_*^{(k_1+\ldots+k_n)}$ from Proposition 3.1 (i). Moreover, we see that

the elements $\bar{x}_1\cdots\bar{x}_n\omega_1^{k_1}\cdots\omega_n^{k_n}$ $(k_1+\cdots+k_n+n = i)$ and $[\Lambda]\omega_1^{l_1}\cdots\omega_n^{l_n}$ $(l_1+\cdots+l_n = i)$ are linearly independent in $HH_*^{(i)}$ when $i > n$. In the case $i \le n$, we can deduce that the elements $\bar{x}_{j_1}\cdots\bar{x}_{j_i}$ $(1 \le j_1 < \ldots < j_i \le n)$ and $[\Lambda]\omega_1^{l_1}\cdots\omega_n^{l_n}$ $(l_1+\cdots+l_n = i)$ are linearly independent in $HH_*^{(i)}$. This completes the proof. □

THEOREM 5.3. *Suppose $m = n$. Let $[\Lambda]$ be the fundamental class of the algebra Λ ([**45**]). If ρ_t is the element of the greatest degree in the regular sequence $\rho_1,\ldots,\rho_n$, then for all i, $HH_j^{(i)} = 0$ whenever $j > \deg[\Lambda]+i(\deg\rho_t-2)$. Moreover $\dim_{\mathbf{Q}} HH^{(i)}_{\deg[\Lambda]+i(\deg\rho_t-2)} = 1$.*

PROOF. By the same argument as the proof of Theorem 3.1, we see that $[\Lambda]\omega_t^i$ represents a non-zero element of $HH_*^{(i)}$. Any element u of $Ann_A(d(\omega_{i_1}),...,d(\omega_{i_s}))\cdot\omega_1^{k_1}\cdots\omega_m^{k_m}$ can be written as $u = (\sum_l a_l b_l)\cdot\omega_1^{k_1}\cdots\omega_m^{k_m}$ with monomials $a_l \in \Lambda$ and $b_l \in \Lambda(\bar{x}_1,\ldots\bar{x}_n)$. Since the algebra Λ is a finite dimensional vector space, it follows that $\deg x_i < \deg\rho_t$ for any i. Therefore $\deg\bar{x}_i \le \deg\rho_t - 2 = \deg\omega_t$. So $\deg(b_l\omega_1^{k_1}\cdots\omega_n^{k_n}) \le \deg\omega_t^i$ when $b_l = \bar{x}_{j_1}\cdots\bar{x}_{j_s}$ and $k_1+\cdots+k_n+s = i$. The fact that $\Lambda^k = 0$ for any $k > \deg[\Lambda]$ enables us to conclude that, $HH_j^{(i)} = 0$ whenever $j > \deg([\Lambda]\omega_t^i)$. Moreover, since $\Lambda^{deg[\Lambda]}$ is a 1-dimensional vector space generated by $[\Lambda]$ ([**45**]), it follows that $HH^{(i)}_{\deg[\Lambda]+i(deg\rho_t-2)}$ is generated by the element $[\Lambda]\omega_t^i$. □

EXAMPLE 5.4. *The minimal model $\mathcal{M}(X)$ of the rational de Rham complex $(\Omega^*(X),\partial)$ for $X = U(2+2)/U(2)\times U(2)$ is given as*

$$\mathcal{M} = (\mathbf{Q}[c_1,c_2]\otimes\Lambda(\tau_1,\tau_2),d),$$

*where $\deg c_i = 2i$, $\deg\tau_j = 2\cdot 2+2j-1$, $d(c_i) = 0$, $d(\tau_1) = \rho_1 = 2c_1c_2 - c_1^3$ and $d(\tau_2) = \rho_2 = c_2^2 - 3c_1^2c_2 + c_1^4$. (see [**25**, Lemma 2.3].) Since*

$$d(\bar{\tau}_1) = \frac{\partial\rho_1}{\partial c_1}\bar{c}_1 + \frac{\partial\rho_1}{\partial c_2}\bar{c}_2 = (2c_2 - 3c_1^2)\bar{c}_1 + 2c_1\bar{c}_2$$

in $\mathcal{K}$, it follows that the element $v = c_1^2\bar{c}_1 - c_1\bar{c}_2$ belongs to $Ann(d\bar{\tau}_1)$. We can see that $v\bar{\tau}_1^{i-1}(i \ge 1)$ is non-zero element of $HH_*^{(i)}$ by degree reasons. Indeed, suppose that $v\bar{\tau}_1^{i-1}$ is zero in $HH_*^{(i)}$. Then we can write $v\bar{\tau}_1^{i-1} = d(\sum_{j=0}^i a_j\bar{\tau}_1^{i-j}\bar{\tau}_2^j)$ for some $a_j \in \mathbf{Q}[c_1,c_2]/(\rho_1,\rho_2)\otimes\wedge(\bar{c}_1,\bar{c}_2)$ in general. Since $\deg v = \deg d(\bar{\tau}_1) < \deg d(\bar{\tau}_2)$, it follows that $v\bar{\tau}_1^{i-1} = d(a_0\bar{\tau}_1^i)$ and $\deg(a_0) = 0$. Therefore we have $c_1^2\bar{c}_1 - c_1\bar{c}_2 = (2a_0c_2 - 3a_0c_1^2)\bar{c}_1 + 2a_0c_1\bar{c}_2$

in $\mathbf{Q}[c_1, c_2]/(\rho_1, \rho_2) \otimes \wedge(\bar{c}_1, \bar{c}_2)$, which is a contradiction. Thus we can conclude that $v\bar{\tau}_1^{i-1} \neq 0$ in $HH_*^{(i)}$.

In particular, $v\bar{\tau}_1^{i-1}$ $(i \geq 1)$ *is different from a linear combination of the elements which we have chosen in the proof of Theorem 5.2 (ii). Thus we have from Theorem 5.2 (ii),*

$$\dim_{\mathbf{Q}} HH_*^{(i)} > 2i \quad \text{for } i > 2,$$
$$\dim_{\mathbf{Q}} HH_*^{(2)} > 4 \quad \text{and}$$
$$\dim_{\mathbf{Q}} HH_*^{(1)} > 4.$$

Since the degree of the fundamental class of $H^*(X; \mathbf{Q})$ *is 8, from Theorem 3.2, we have* $\dim_{\mathbf{Q}} HH_{8+6i}^{(i)} = 1$ *and* $HH_j^{(i)} = 0$ *for* $j > 8 + 6i$.

CHAPTER 2

The vanishing problem of the string class with degree 3

In this chapter, for a (differential) manifold M, we also denote the space of all smooth maps from the circle S^1 into M by LM.In the case, in the square (3.3) of Chapter 0, M^I means the space of all smooth maps from the interval $[0, 1]$ into M.

1. String class

Let M be a (differentiable) manifold and G a Lie group.

DEFINITION 1.1 ([**24**]). *A (differentiable) principal bundle over* M *with group* G *consists of a manifold* P *and an action of* G *on* P *satisfying the following conditions:*

(1) G *acts freely on* P *on the right:* $(u, a) \in P \times G \to u \cdot a \in P$,

(2) M *is the quotient space of* P *by the equivalence relation induced by* G, $M = P/G$, *and the canonical projection* $\pi : P \to M$ *is differentiable, and*

(3) P *is locally trivial, that is, every point* x *of* M *has a neighborhood* U *such that* $\pi^{-1}(U)$ *is isomorphic with* $U \times G$.

Recall the special orthogonal group $SO(n) = \{A \in M(n, \mathbf{R}); A^tA = I, \det A = 1\}$, where $M(n, \mathbf{R})$ is the set of all matrices of degree n with entries in $\mathbf{R}$ and the spinor group $Spin(n)$, which is a Lie group and also a universal covering group of $SO(n)$ for $n \geq 3$.

Let ξ be an $SO(n)$-principal bundle $P \to M$ over a simply connected manifold M for $n \geq 3$.

DEFINITION 1.2 ([**36**]). *The bundle* ξ *is said to have spin structure if the structure group of* P *lifts to* $Spin(n)$ *i.e., there is a commutative diagram:*

$$\begin{array}{ccccc} Spin(n) & \longrightarrow & Q & \longrightarrow & M \\ \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle =} \\ SO(n) & \longrightarrow & P & \xrightarrow{\pi} & M, \end{array} \qquad (1.1)$$

where ϕ_0 is the universal covering homomorphism with the kernel $\{-1,1\} \cong \mathbf{Z}/2$ and ϕ satisfies $\phi(xg) = \phi(x)\phi_0(g)$ for any $x \in Q$ and any $g \in Spin(n)$.

It is known that $LSpin(n) \to LQ \to LM$ is also a principal $LSpin(n)$-bundle [**39**]. Assume that ξ has a spin structure $Q \to M$ for $n \geq 5$ and let $S^1 \to \widehat{LSpin(n)} \to LSpin(n)$ the universal central extension of $LSpin(n)$ by the circle S^1 ([**39**]).

DEFINITION 1.3 ([**36**]). *The bundle ξ is said to have string structure if the structure group of LQ lifts to $\widehat{LSpin(n)}$.*

DEFINITION 1.4 ([**36**]). *We denote the pull back of the generator of $H^3(LBSpin(n);\mathbf{Z})$ by the classifying map of $L\pi : LQ \to LM$ as $\mu(Q)$ and call it string class.*

Then it is known that the string class $\mu(Q)$ in $H^3(LM;\mathbf{Z})$ is an obstruction to lift the structure group of the $LSpin(n)$-bundle $LQ \to LM$ to $\widehat{LSpin(n)}$ ([**36**]). Let $p_1(\xi)$ be the first Pontrjagin class of ξ ([**24**]). It is known that $p_1(\xi)$ is two times the pullback of the generator ι of $H^4(BSpin(n);\mathbf{Z})$ by the classifying map of a spin structure $Q \to M$ for ξ ([**36**, Lemma 2.2]). Following [**36**], we denote the pullback of ι by $\frac{1}{2}p_1(\xi)$. Also the map $\int_{S^1} \circ ev^* : H^*(X;\mathbf{Z}) \to H^{*-1}(LX;\mathbf{Z})$ will be denoted by $\mathcal{D}_X$ and called *the D-map of X*, where $\int_{S^1} : H^*(S^1 \times LX;\mathbf{Z}) \to H^{*-1}(LX;\mathbf{Z})$ is the integration map along S^1 and $ev : S^1 \times LX \to X$ is the evaluation map, $ev(\gamma, t) = \gamma(t)$ for $\gamma \in LX$ and $t \in S^1$. The argument of the proof of [**36**, Theorem 3.1] enables us to

LEMMA 1.5 ([**36**]). $\mathcal{D}_M(\frac{1}{2}p_1(\xi)) = \mu(Q)$.

DEFINITION 1.6. *The D-map of M is said* good *if $\mathcal{D}_M$ is a monomorphism.*

Therefore if the D-map of M is good, then $\mu(Q)$ vanishes if and only if $\frac{1}{2}p_1(\xi)$ vanishes for any $SO(n)$ -bundle ξ with a spin structure $Q \to M$. In this case we can deduce that the $LSpin(n)$ -bundle over the infinite dimensional manifold LM has a string structure if and only if $1/2$ the first Pontrjagin class of the $SO(n)$-bundle over the finite dimensional manifold M vanishes. Our goal is to study which manifolds have a good D-map. McLaughlin showed

THEOREM 1.7 ([**36**, Theorem 3.1]). *The D-map of any 2-connected manifold is good.*

Also K.Kuribayashi showed

THEOREM 1.8 ([**26**, Theorem 1]). *Let M be a simply connected manifold. If $H^4(M;\mathbf{Z})$ is torsion free and dim $H^2(M;\mathbf{R}) \leq 1$. Then the D-map of M is good. Therefore, in this case, $\frac{1}{2}p_1(\xi)$ vanishes if the string class $\mu(Q)$ vanishes.*

We can deduce from Theorem 1.8 that the complex Grassman manifold has the good D-map.

REMARK 1.9. *So far, we have considered the string class of an $SO(n)$-bundle in the case where $n \geq 5$. The case $n = 4$ must be treated separately as mentioned in* [**36**, Remark, page 150] *because the universal central extension of $LSpin(n)$ is an extension by a 2-torus. The fact that $SO(4)$ is not simple causes the difference. In the case where $n = 3$, since SO(3) is simple, we can define the string class of an $SO(3)$-bundle with a spin structure in similar fashion to the case $n \geq 5$. However, the index of the homomorphism $B\pi^*$: $H^4(BSO(3);\mathbf{Z}) = \mathbf{Z} \to H^4(BSpin(3);\mathbf{Z}) = \mathbf{Z}$ is 4, not 2, where $\pi : Spin(3) \to SO(3)$ is the universal covering. This fact is proved by using the same argument as the proof of* [**36**, Lemma 2.2, page 148]. *Notice that $H^5(BSO(3);\mathbf{Z})$ is zero though $H^5(BSO(n);\mathbf{Z}) = \mathbf{Z}/2$ for $n \geq 5$. Thus the string class $\mu(Q)$ of an $SO(3)$-bundle ξ with a spin structure $Q \to M$ can be regarded as the image of 1/4 the Pontrjagin class of ξ by the D-map $\mathcal{D}_M : H^4(M;\mathbf{Z}) \to H^3(LM;\mathbf{Z})$.*

In Section 3, we generalize Theorem 1.8 (see Theorem 3.4).

2. Hochschild homology below degree 3

Let X be a path-connected and simply connected space. In order to consider the algebra structure of $H^*(LX;\mathbf{Z}/p)$, we use the Eilenberg Moore spectral sequence converging to $H^*(LX;\mathbf{Z}/p)$ whose E_2-term is isomorphic to the Hochschild homology of $H^*(X;\mathbf{Z}/p)$. Before we begin calculating this spectral sequence, we give an available complex to determine the algebra structure of the Hochschild homology of a certain commutative algebra. A commutative algebra A will mean a positive graded commutative algebra over $\mathbf{Z}/p$ satisfying that $A^0 = \mathbf{Z}/p$ and $A^1 = 0$.

Let Λ be a commutative algebra $\Lambda(y_1, ..., y_l) \otimes \mathbf{Z}/p[x_1, .., x_n]/(\rho_1, .., \rho_m)$, where ρ_i is decomposable for any i. We will suppose that $2 \leq \deg x_1 \leq \cdots \leq \deg x_n$, $3 \leq \deg y_1 \leq \cdots \leq \deg y_l$, $\deg \rho_1 \leq \cdots \leq \deg \rho_m$ and $l = 0$ if $p = 2$. If $\rho_1, .., \rho_m$ is a regular sequence, the Koszul-Tate complex of Λ in Section 2 of Chapter 1 is a complex for computing the Hochschild homology $HH_*(\Lambda)$. In the general case, we can also obtain a complex for computing $HH_*(\Lambda)$ by extending the Koszul-Tate complex.

PROPOSITION 2.1. *The Hochschild homology of* Λ *is calculable as the homology of the following complex* $(\mathcal{E}, d)$ *:*

$$\mathcal{E} := \Lambda \otimes \Gamma[\bar{y}_1, .., \bar{y}_l] \otimes \Lambda(\bar{x}_1, .., \bar{x}_n) \otimes \Gamma[\omega_1, .., \omega_m] \otimes \mathcal{C},$$

$d(\omega_i) = \sum_{j=1}^{n} \frac{\partial \rho_i}{\partial x_j} \bar{x}_j$, $d(\lambda) = d(\bar{y}_i) = d(\bar{x}_j) = 0$ *for* $\lambda \in \Lambda$, $i = 1, .., l$, $j = 1, .., n$ *and* bideg $\lambda = (0, \deg \lambda)$ *for* $\lambda \in \Lambda$, bideg $\bar{x}_j = (-1, \deg x_j)$, bideg $\bar{y}_i = (-1, \deg y_i)$, bideg $\omega_i = (-2, \deg \rho_i)$. *Here* $\mathcal{C}$ *is a suitable differential graded algebra which is a tensor product of an exterior algebra and a divided power algebra. Moreover, the differential* d *satisfies that*

$$\{d\mathcal{E} \cap \Lambda \otimes \Gamma[\bar{y}_1, .., \bar{y}_l] \otimes \Lambda(\bar{x}_1, .., \bar{x}_n)\}^{\leq 3} =$$
$$\{d(\Lambda \otimes \Gamma[\bar{y}_1, .., \bar{y}_l] \otimes \Lambda(\bar{x}_1, .., \bar{x}_n) \otimes \Gamma[\omega_1, .., \omega_m]) \cap \Lambda \otimes \Gamma[\bar{y}_1, .., \bar{y}_l] \otimes \Lambda(\bar{x}_1, .., \bar{x}_n)\}^{\leq 3}.$$

PROOF. Let A and B denote the commutative algebra $\Lambda(y_1, .., y_l)$ and $\mathbf{Z}/p[x_1, ..x_n]/(\rho_1, .., \rho_m)$ respectively. If $\rho_1, .., \rho_m$ is a regular sequence, then there exists the following proper projective resolution $\mathcal{F} \xrightarrow{\mu} B \longrightarrow 0$ of B as a left $B \otimes B$-module ([**44**], [**25**, Proposition 1.1]):

$$\mathcal{F} = B \otimes B \otimes \Lambda(\bar{x}_1, .., \bar{x}_n) \otimes \Gamma[\omega_1, .., \omega_m],$$

$\mu : B \otimes B \to B$ is the multiplication of B, $d(\bar{x}_j) = x_j \otimes 1 - 1 \otimes x_j$ and $d(\omega_i) = \sum_{j=1}^{n} \zeta_{ij} \bar{x}_j$, where ζ_{ij} is an element in $B \otimes B$ satisfying $\rho_i \otimes 1 - 1 \otimes \rho_i = \sum_{j=1}^{n} \zeta_{ij}(x_j \otimes 1 - 1 \otimes x_j)$ and $\mu(\zeta_{ij}) = \partial \rho_i / \partial x_j$.

In particular, we can choose the element $\sum_{k=1}^{n} \mu_{jk}^{(i)} x_k \otimes 1 + \sum_{k=1}^{n} 1 \otimes \mu_{kj}^{(i)} x_k$ as the element ζ_{ij} mentioned above if $\rho_i = \sum_{j=1}^{n} \sum_{k=1}^{n} \mu_{jk}^{(i)} x_k x_j$.

Let us consider the general case where $\rho_1, .., \rho_m$ are decomposable elements. By modifying the method to construct a minimal model of a differential graded algebra, we obtain the required differential graded algebra $\mathcal{E}$. The argument of [**44**, Lemma 3.3] enables us to deduce that $H^{0,*}(\mathcal{F}) = 0$. When $i + j = 1$, every element in $\mathcal{F}^{i,j}$ can be written by a linear combination of elements $\bar{x}_1, .., \bar{x}_n$. Therefore, $H^{i,j}(\mathcal{F}) = 0$ for $i + j = 1$. Suppose that $i + j = 2$ and $H^{i,j}(\mathcal{F}) \neq 0$. Then, from the definition of the differential d, we obtain $(i, j) = (-2, 4)$. If the element $u = \sum_{i<j} a_{ij} \bar{x}_i \bar{x}_j$ is in Ker $d^{-2,4}$, then $0 = du = \sum_{i<j} a_{ij}(x_i \otimes 1 - 1 \otimes x_i)\bar{x}_j - \sum_{i<j} a_{ij}(x_j \otimes 1 - 1 \otimes x_j)\bar{x}_i$. Thus we see that $a_{1n}(x_1 \otimes 1 - 1 \otimes x_1) + \cdots + a_{n-1\ n}(x_{n-1} \otimes 1 - 1 \otimes x_{n-1}) = 0$ and hence $a_{in} = 0$ for any $i < n$. Inductively, we have $a_{ij} = 0$ for any i and j. From this fact, we can conclude that each element of a basis $\{z_1, .., z_s\}$ for $H^{-2,4}(\mathcal{F})$ represents an element $\sum a_{ij} \bar{x}_i \bar{x}_j + \sum b_k \omega_k$, where b_k is nonzero for some k. The element z_t and its representative element will be denoted by the same notation. We define the differential graded algebra $(\mathcal{F}_1, d)$ by $\mathcal{F}_1 = \mathcal{F} \otimes \Lambda(\bar{z}_t)$ and $d(\bar{z}_t) = z_t$, where bideg $z_t = (-3, 4)$. Clearly, $H^{i,j}(\mathcal{F}_1) = 0$ for $i + j = 2$. From the form of a representative element of z_t, it follows that $d(\Lambda(\bar{z}_t)) \cap B \otimes B \otimes \Lambda(\bar{x}_1, .., \bar{x}_n) = 0$. Consider the case where $i + j = 3$. It is easy to verify that $\mathrm{Ker}\, d^{-2,5} \cap \mathbf{Z}/p\{\bar{x}_i \bar{x}_j; 1 \leq i, j \leq n\} = 0$ in the only case where $p = 2$ and that $\mathrm{Ker}\, d^{-3,6} \cap \mathbf{Z}/p\{\bar{x}_i \bar{x}_j \bar{x}_k; 1 \leq i, j, k \leq n\} = 0$. We define the elements $\bar{v}_\beta$ with total degree 3 corresponding to representative elements v_β of a basis of $H^{i,j}(\mathcal{F}_1)$ for $(i, j) = (-2, 5)$ and $(-3, 6)$. Put $\mathcal{F}_2 = \mathcal{F}_1 \otimes \Gamma[\bar{v}_\beta]$ and extend the differential d by demanding that $d(\bar{v}_\beta) = v_\beta$. The elements of $\mathrm{Ker}\, d \cap \mathcal{F}_1^{-1,4}$ are characterized as follows:

LEMMA 2.2. *Let* u *be an element of* $\mathrm{Ker}\, d \cap \mathcal{F}_1^{-1,4}$. *Then* u *can be written as* $\sum_{j=1}^{n}(\sum_{k=1}^{n} \lambda_{jk} x_k \otimes 1 + \sum_{k=1}^{n} 1 \otimes \lambda_{kj} x_k)\bar{x}_j$ *with coefficients* λ_{jk} *satisfying that* $\lambda_{jj} = t^{(1)}\mu_{jj} + \cdots + t^{(m)}\mu_{jj}$ *and* $(\lambda_{jk} + \lambda_{kj}) = t^{(1)}(\mu_{jk}^{(1)} + \mu_{kj}^{(1)}) + \cdots + t^{(m)}(\mu_{jk}^{(m)} + \mu_{kj}^{(m)})$ *for some* $t^{(i)}$ *(for the notation* $\mu_{jk}^{(l)}$ *see the definition of the above resolution* $\mathcal{F}$*). Therefore, the element* $1 \otimes_{\Lambda \otimes \Lambda} u$ *in* $\Lambda \otimes_{\Lambda \otimes \Lambda} \mathcal{F}_1$ *belongs to* $\mathbf{Z}/p\{d(\omega_1), .., d(\omega_m)\}$.

Let $\{u_t\}$ be a basis for $H^{-1,4}(\mathcal{F}_1)$. We extend the complex $\mathcal{F}_2$ to $\mathcal{F}_3 = \mathcal{F}_2 \otimes \Gamma[\bar{u}_t] = \mathcal{F}_1 \otimes \Gamma[\bar{v}_\beta] \otimes \Gamma[\bar{u}_t]$ with the differential defined by $d(\bar{u}_t) = u_t$. From this construction, we see that $H^{i,j}(\mathcal{F}_2) = 0$ for $i + j = 3$ and $d(\Gamma[\bar{v}_\beta]) \cap B \otimes B \otimes \Lambda(\bar{x}_1, .., \bar{x}_n) = 0$. By continuing the same process above total degree 4, we can get a proper projective resolution $\tilde{\mathcal{E}}_B$ of B as a $B \otimes B$-module : $\tilde{\mathcal{E}}_B = \mathcal{F} \otimes \mathcal{C}$. By virtue of [**44**, Lemma 3.2], we conclude that the differential graded algebra $(\widetilde{\mathcal{E}}_A, d)$, defined by $\widetilde{\mathcal{E}}_A = A \otimes A \otimes \Gamma[\bar{y}_1, .. \bar{y}_l]$ and $d(\bar{y}_i) = y_i \otimes 1 - 1 \otimes y_i$, is a proper projective resolution of A as an $A \otimes A$-module. Therefore, the differential graded algebra $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}_A \otimes \widetilde{\mathcal{E}}_B$ is a proper projective resolution of Λ as a $\Lambda \otimes \Lambda$-module. Thus the Hochschild homology $HH_{*,*}(\Lambda) = Tor^{*,*}_{\Lambda\otimes\Lambda}(\Lambda, \Lambda)$ is obtained as the homology of the complex $(\mathcal{E}, d) = (\Lambda \otimes_{\Lambda\otimes\Lambda} \widetilde{\mathcal{E}}, 1 \otimes d)$. From Lemma 2.2, it follows that $d(\bar{u}_t) = 1 \otimes_{\Lambda\otimes\Lambda} d(\bar{u}_t) = 1 \otimes_{\Lambda\otimes\Lambda} u_t$ is in $\mathbf{Z}/p\{d(\omega_1), .., d(\omega_m)\}$. This fact and the definitions of $d(\bar{z}_t)$ and $d(\bar{v}_\beta)$ allow us to deduce that

$$\{d\mathcal{E} \cap \Lambda \otimes \Gamma[\bar{y}_1, .., \bar{y}_l] \otimes \Lambda(\bar{x}_1, .., \bar{x}_n)\}^{\leq 3} =$$

$$\{d(\Lambda \otimes_{\Lambda\otimes\Lambda} (\widetilde{\mathcal{E}}_A \otimes \mathcal{F})) \cap \Lambda \otimes \Gamma[\bar{y}_1, .., \bar{y}_l] \otimes \Lambda(\bar{x}_1, .., \bar{x}_n)\}^{\leq 3}.$$

This completes the proof of Proposition 2.1. □

PROOF OF LEMMA 2.2. For any element $u \in \operatorname{Ker} d \cap \mathcal{F}_1^{-1,4}$, we can write $u = \sum_j \xi_j \bar{x}_j$, where $\xi_j = \sum_{k=1}^n \lambda_{jk} x_k \otimes 1 + \sum_{k=1}^n 1 \otimes \lambda'_{jk} x_k$. Since $d(u) = 0$ in $\mathcal{F}_1^{0,4} = \mathcal{F}^{0,4}$, it follows that the element

$$du = \sum_{j=1}^n \sum_{k=1}^n \lambda_{jk} x_k x_j \otimes 1 - \sum_{j=1}^n \sum_{k=1}^n \lambda_{jk} x_k \otimes x_j + \sum_{j=1}^n \sum_{k=1}^n \lambda'_{jk} x_j \otimes x_k - \sum_{j=1}^n \sum_{k=1}^n 1 \otimes \lambda'_{jk} x_k x_j$$

belongs to the ideal $(\rho_i \otimes 1, 1 \otimes \rho_i; 1 \leq i \leq m)$ in $\mathbf{Z}/p[x_1, .., x_n] \otimes \mathbf{Z}/p[x_1, .., x_n]$. This fact enables us to conclude that $\sum_{j=1}^n \sum_{k=1}^n \lambda_{jk} x_k \otimes x_j = \sum_{j=1}^n \sum_{k=1}^n \lambda'_{jk} x_j \otimes x_k$ and so $\lambda_{jk} = \lambda'_{kj}$. Moreover we see $\sum_{j=1}^n \sum_{k=1}^n \lambda_{jk} x_k x_j = t^{(1)} \rho_1 + \cdots + t^{(m)} \rho_m = t^{(1)}(\sum_{j=1}^n \sum_{k=1}^n \mu_{jk}^{(1)} x_k x_j) + \cdots + t^{(m)}(\sum_{j=1}^n \sum_{k=1}^n \mu_{jk}^{(m)} x_k x_j)$ in $\mathbf{Z}/p[x_1, .., x_n]$. Thus we have

$$\begin{aligned}
\sum_{j=1}^n \lambda_{jj} x_j^2 + \sum_{j<k} (\lambda_{jk} + \lambda_{kj}) x_k x_j &= t^{(1)}(\sum_{j=1}^n \mu_{jj}^{(1)} x_j^2 + \sum_{j<k} (\mu_{jk}^{(1)} + \mu_{kj}^{(1)}) x_k x_j) + \cdots \\
&\quad + t^{(m)}(\sum_{j=1}^n \mu_{jj}^{(m)} x_j^2 + \sum_{j<k} (\mu_{jk}^{(m)} + \mu_{kj}^{(m)}) x_k x_j)
\end{aligned}$$

in $\mathbf{Z}/p[x_1, .., x_n]$. Therefore, the required relations for λ_{jk} are obtained. Let μ be the multiplication of B. Since $\mu(\zeta_{ij}) = \sum_{k=1}^n (\mu_{jk}^{(i)} + \mu_{kj}^{(i)}) x_k$, it follows that

$$\begin{aligned}
1 \otimes_{\Lambda\otimes\Lambda} u &= \sum_{j=1}^n (\sum_{k=1}^n (\lambda_{jk} + \lambda_{kj}) x_k) \bar{x}_j \\
&= \sum_{j=1}^n t^{(1)} (\sum_{k=1}^n (\mu_{jk}^{(1)} + \mu_{kj}^{(1)}) x_k) \bar{x}_j + \cdots + \sum_{j=1}^n t^{(m)} (\sum_{k=1}^n (\mu_{jk}^{(m)} + \mu_{kj}^{(m)}) x_k) \bar{x}_j \\
&= \sum_{j=1}^n t^{(1)} \mu(\zeta_{1j}) \bar{x}_j + \cdots + \sum_{j=1}^n t^{(m)} \mu(\zeta_{mj}) \bar{x}_j \\
&= t^{(1)} d(\omega_1) + \cdots + t^{(m)} d(\omega_m)
\end{aligned}$$

Thus we have Lemma 2.2. □

Applying Proposition 2.1, we can partially know the algebra structure of the Hochschild homology of the graded algebra Λ.

PROPOSITION 2.3. *Let Λ be the graded algebra $\Lambda(y_1, .. y_l) \otimes \mathbf{Z}/p[x_1, .. x_n]/(\rho_1, .., \rho_m)$. Then there exists a morphism of algebras*

$$\phi : \Lambda(y_1, .., y_l) \otimes \Gamma[\bar{y}_1, .., \bar{y}_l] \otimes \{A/(d\omega_1, .., d\omega_m)_A\} \to HH_{*,*}(\Lambda)$$

which is a monomorphism below total degree 3, where $A = \mathbf{Z}/p[x_1, .. x_n]/(\rho_1, .., \rho_m) \otimes \Lambda(\bar{x}_1, .., \bar{x}_n)$.

PROOF. Let us consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{K} \otimes \mathbf{Z}/p\{\omega_1, .., \omega_m\} & \xrightarrow{\bar{d}} & \mathcal{K} := \Lambda \otimes \Gamma[\bar{y}_1, .., \bar{y}_l] \otimes \Lambda(\bar{x}_1, .., \bar{x}_n) & \longrightarrow & 0 \\
\downarrow j & & \downarrow i & & \downarrow \\
\mathcal{E} & \xrightarrow[d]{} & \mathcal{E} & \xrightarrow{d} & \mathcal{E}
\end{array}$$

where i and j are the inclusion maps and $\bar{d}$ is the restriction of d to $\mathcal{K} \otimes \mathbf{Z}/p\{\omega_1, .., \omega_m\}$. Suppose that $H(i)[z] = 0$ and the total degree of $[z]$ is below 3. Then $i(z)$ belongs to $\{d(\mathcal{E}) \cap \mathcal{K}\}^{\leq 3}$. From Proposition 2.1, we can see that $i(z)$ is in the vector space $\{\operatorname{Im} d \circ j\}^{\leq 3}$. Therefore z is an element in $\operatorname{Im} \bar{d}$. The map $\phi = H(i)$ is the demanded homomorphism. □

We consider the ring structure of $H^*(LX; \mathbf{Z}/p)$, in particular, to clarify indecomposable elements with degree below 3 in $H^*(LX; \mathbf{Z}/p)$ and relations between their elements in

$H^3(LX;\mathbf{Z}/p)$. To this end, we use the Eilenberg Moore spectral sequence converging to $H^*(LX;\mathbf{Z}/p)$ ([**44**],[**25**]) whose E_2-term is the Hochschild homology of $H^*(X;\mathbf{Z}/p)$:

$$E_2^{*,*} \cong Tor^{*,*}_{H^*(X;\mathbf{Z}/p)\otimes H^*(X;\mathbf{Z}/p)}(H^*(X;\mathbf{Z}/p), H^*(X;\mathbf{Z}/p)) = HH_*(H^*(X;\mathbf{Z}/p)).$$

Proposition 2.3 plays an important rule in explaining the algebra structure of $H^*(LX;\mathbf{Z}/p)$.

Notation. Let Λ be a graded algebra and S a subset of Λ. Then the ideal of Λ generated by elements of S will be denoted by $(S)_\Lambda$ as in Chapter 1. For any graded vector space $V = \oplus_{i\geq 0}V^i$, $V^{\leq n}$ means $\oplus_{0\leq i\leq n}V^i$. We denote the commutative algebra with the 2-simple system of generators $\{z_j\}_{j=1,..,n}$ by $\Delta(z_1,..,z_n)$. Let T be a subset of a vector space W over a field $\mathbf{k}$. We denote the subspace of W generated by elements of T by $\mathbf{k}\{T\}$.

THEOREM 2.4. *Suppose that X is a simply connected space and that there exists a morphism of algebras*

$$\phi : B = \Lambda(y_1,..y_l)\otimes \mathbf{Z}/p[x_1,..x_n]/(\rho_1,..,\rho_m) \to H^*(X;\mathbf{Z}/p),$$

which is an isomorphism below degree 4, where $\rho_1,..,\rho_n$ are decomposable elements with degree 4, $\deg x_j = 2$ or 4, $\deg y_i = 3$ and $l = 0, \deg x_j = 2, 3$ or 4 if $p = 2$. We regard $H^*(LX;\mathbf{Z}/p)$ as a B-module via the composition map $\pi^*\phi$. Then there exists a morphism of algebras and of B-modules

$$\psi : \Lambda(y_1,..y_l)\otimes\Gamma[\bar{y}_1,..,\bar{y}_l]\otimes\{A_p/(d(\omega_1),..,d(\omega_m))_{A_p}\} \to H^*(LX;\mathbf{Z}/p)$$

which is a monomorphism below degree 3, where $d(\omega_i) = \sum_{j=1}^{n}\frac{\partial\rho_i}{\partial x_j}\bar{x}_j$, $\deg\bar{x}_j = \deg x_j - 1$, $\deg\bar{y}_i = 2$, $\deg\omega_i = \deg\rho_i - 2$ and $A_p = \mathbf{k}_p[x_1,..,x_n]/(\rho_1,..,\rho_m)\otimes\Lambda(\bar{x}_1,..,\bar{x}_n)$ if $p\neq 2$, and $A_2 = \mathbf{k}_2[x_1,..,x_n]/(\rho_1,..,\rho_m)\otimes\Delta(\bar{x}_1,..,\bar{x}_n)$ if $p = 2$, where $\Delta(\bar{x}_1,..,\bar{x}_n)$ is the commutative algebra with the 2-simple system of generators $\{\bar{x}_j\}_{j=1,..,n}$.

REMARK 2.5. *In the case where $p = 2$, we cannot solve extension problems completely by using the usual argument on total degrees and column degrees of the associated bigraded algebra $E_0^{*,*}$. For example, it may be possible that $\bar{x}_i^2$ is equal to some $\bar{y}_j$. However some information about the squaring operations in $H^*(X;\mathbf{Z}/2)$ allows us to determine whether or not $\bar{x}_i^2$ is equal to $\bar{y}_j$. To be exact, if $Sq^1x_i = \varepsilon y_j$ then $\bar{x}_i^2 = \varepsilon\bar{y}_j$, where $\varepsilon = 0$ or 1. (see the proof of Chapter 1, Theorem 4.8)*

Before we prove Theorem 2.4, we prepare a lemma.

LEMMA 2.6. *Let C_1 and C_2 be commutative algebras. Suppose that there exists a morphism of algebras $\theta : C_1 \to C_2$ which is an isomorphism below degree s. Then $Tor_{\theta\otimes\theta}(\theta,\theta) : Tor^{i,j}_{C_1\otimes C_1}(C_1,C_1) \to Tor^{i,j}_{C_2\otimes C_2}(C_2,C_2)$ is an isomorphism if $i = 0$ and $j\leq s$, $i = -1$ and $j \leq s-1$ or $i < -1$ and $i+j < s-i-2$.*

PROOF. Let $Bar^{*,*}(C_1)$ be the complex obtained from the bar resolution of C_1 as an $C_1\otimes C_1$-module and $Bar^{*,*}(C_2)$ the similar complex constructed from C_2. Let t be an element of $Bar^{-i,*}(C_1)$. We can write $u = a[b_1|\cdots|b_i]c$, where a and c are elements of A and b_l is an element of $\overline{C_1\otimes C_1}$. If there exists an element b_l such that $\deg b_l > s$, then total deg$t$ = $\deg a + \deg b_1 + \cdots + \deg b_l + \cdots + \deg b_i + \deg c - i > 0 + 2 + \cdots + 2 + s + 2 + \cdots + 2 + 0 - i = s + i - 2$ when $i \neq 0$. Thus we see that the morphism $Bar^{i,j}(\theta) : Bar^{i,j}(C_1) \to Bar^{i,j}(C_2)$, which is induced from θ, is an isomorphism if $-i + j \leq s + i - 2$ and $i\neq 0$. It is clear that $B^{0,j}(\theta)$ is an isomorphism if $j \leq s$. Therefore we have Lemma 2.6. □

PROOF OF THEOREM 2.4. Let $\{E_r, d_r\}$ be the Eilenberg Moore spectral sequence converging to $H^*(LX;\mathbf{Z}/p)$. By virtue of Proposition 2.3 and Lemma 2.6, we have a homomorphism ψ from $B := \Lambda(y_1,..y_l)\otimes\Gamma[\bar{y}_1,..,\bar{y}_l]\otimes\{A_p/(d(\omega_1),..,d(\omega_m))_{A_p}\}$ to $E_2^{*,*}$ which is a monomorphism below degree 3. From [**42**, Proposition 4.2], it is seen that $\pi^*(y_j) = y_j \in F^0H^*(LX;\mathbf{Z}/p)$ and $\pi^*(x_i) = x_i \in F^0H^*(LX;\mathbf{Z}/p)$. Therefore the injectivity of π^* allows us to conclude that $d_2 : E_2^{-2,*} \to E_2^{0,*}$ is trivial. Since $E_2^{i,j} = 0$ if $q < -2p$, it follows that the elements in $E_2^{-1,j}$ survive in the E_∞-term if $j \leq 4$. Thus we have an monomorphism $\psi : B^{\leq 3} \to (TotE_0^{*,*})^{\leq 3}$. In order to complete the proof of Theorem 2.4, we must solve extension problems below degree 3. More precisely, we need to consider the problem that whether $\sum\lambda_{i,j}x_i\bar{x}_j = 0$ in $H^*(LX;\mathbf{Z}/p)$ when $\sum\lambda_{ij}x_i\bar{x}_j \in \mathbf{Z}/p\{d(\omega_1),..,d(\omega_m)\}$. Note that the element $\bar{x}_j$ in $E_0^{-1,*}$ and its representative element in $H^*(LX;\mathbf{Z}/p)$ are denoted by the same notation. Since the generators with degree 3 and filtration degree 0 are the elements $y_1, .., y_l$, we can write the element $\sum \lambda_{ij} x_i \bar{x}_j$ as $\sum \mu_k y_k$ with some constants μ_k. Let $\pi : LX \to X$ be the fibration defined by $\pi(\gamma) = \gamma(0)$. From [**42**, Proposition 4.2], we see that the element y_k in $E_0^{0,*}$ is identified with the element $\pi^*(y_k)$. Hence the given equality is written as $\sum \lambda_{ij} x_i \bar{x}_j = \sum \mu_k \pi^*(y_k)$. Let s be the section of the fibration $LX \to X$ defined by $s(x) = C_x$, where C_x is the constant loop at x. Since we can choose a representative element of $\bar{x}_i$ so that $s^*(\bar{x}_i) = 0$ in $H^*(X; \mathbf{Z}/p)$, it follows that $\sum \mu_k y_k = s^*(\sum \lambda_{ij} x_i \bar{x}_j) = 0$ in $H^*(X; \mathbf{Z}/p)$ and hence $\mu_k = 0$ for any k. Thus $\sum \lambda_{i,j} x_i \bar{x}_j = 0$ in $H^*(LX; \mathbf{Z}/p)$ when $\sum \lambda_{ij} x_i \bar{x}_j \in \mathbf{Z}/p\{d(\omega_1), .., d(\omega_m)\}$. Therefore we have Theorem 2.4. □

3. Kernel of $\int_{S^1} \circ ev^*$

In this section, we consider the injectivity of the D-map $\mathcal{D}_X : H^4(X; \mathbf{Z}) \to H^3(LX; \mathbf{Z})$. To this end, we study the algebra structure of $H^*(LX; \mathbf{Z}/p)$ and the injectivity of the *mod p D-map* $\mathcal{D}_{X,p} = \int_{S^1} \circ ev^* : H^4(X; \mathbf{Z}/p) \to H^3(LX; \mathbf{Z}/p)$ for any prime p. The behavior of $\mathcal{D}_{X,p}$ on $H^4(X; \mathbf{Z}/p)$ is determined by Theorem 2.4 and the following Theorem 3.1. Here we now note that the D-map $\mathcal{D}_X$ and D-map $\mathcal{D}_{X,p}$ are derivations (see [**27**, Section 3]). More precisely, $\mathcal{D}_X(xy) = \mathcal{D}_X(x)y + (-1)^{\deg x} x \mathcal{D}_X(y)$ for any $x, y \in H^4(X; \mathbf{Z})$ and $\mathcal{D}_{X,p}(xy) = \mathcal{D}_{X,p}(x)y + (-1)^{\deg x} x \mathcal{D}_{X,p}(y)$ for any $x, y \in H^4(X; \mathbf{Z}/p)$, respectively. We often identify the elements y_j and x_i with $\phi(y_j)$ and $\phi(x_i)$ in Theorem 2.4, respectively. Let $\sigma^* : H^*(X; \mathbf{Z}/p) \to H^{*-1}(\Omega X; \mathbf{Z}/p)$ be the cohomology suspension and $i : \Omega X \to LX$ the inclusion map.

THEOREM 3.1. *One can choose the elements $\bar{y}_i$ and $\bar{x}_j$ in Theorem 2.4 so that $i^*(\bar{y}_i) = \sigma^*(y_i)$, $i^*(\bar{x}_j) = \sigma^*(x_j)$ and*

$$\mathcal{D}_{X,p}|_{H^4(X;\mathbf{Z}/p)} = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \bar{x}_j.$$

Notice that, for any simply connected space X, one can construct an algebra and a morphism of algebras ϕ satisfying the condition of Theorem 2.4 by using indecomposable elements x_j and y_i in $H^*(X; \mathbf{Z}/p)$.

PROOF OF THEOREM 3.1. Let $\{E_r, d_r\}$ be the Eilenberg Moore spectral sequence used in the proof of Theorem 2.4. By applying [**42**, Proposition 4.5] and the same argument as the proof of [**25**, Lemma 1.3], we can show that $i^*(\bar{y}_i) = \sigma^*(y_i)$ and $i^*(\bar{x}_j) = \sigma^*(x_j)$ with any choice of representative elements of $\bar{y}_i$ and $\bar{x}_j$ in $E_0^{-1,*}$. To proceed with the proof, we need the following:

LEMMA 3.2. *For each indecomposable element $\bar{x}_j$ in $E_0^{-1,*}$, one can choose its representative element $\bar{x}_j$ in $H^*(LX; \mathbf{Z}/p)$ so that $\mathcal{D}_{X,p}(x_j) = \bar{x}_j$.*

From Lemma 3.2 and the fact that the D-map $\mathcal{D}_{X,p}$ is a derivation, we can get $\mathcal{D}_{X,p} = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \bar{x}_j$. This completes the proof. □

PROOF OF LEMMA 3.2. Let $f_j : X \to K_j = K(\mathbf{Z}/p, n_j)$ be the map representing the element x_j of $H^{n_j}(X; \mathbf{Z}/p)$, where $n_j = \deg x_j$. From the naturality of the D-map $\mathcal{D}_{X,p}$, to prove Lemma 3.2, it suffices that $\mathcal{D}_{K_j,p}(\iota_j) = \bar{\iota}_i$, where ι_j is the fundamental element of $H^{n_j}(K_j; \mathbf{Z}/p)$ and $\bar{\iota}_j$ is the element corresponding to ι_j (see Theorem 2.4). We denote

$$\tilde{\mathcal{D}} = \int_{S^1} \circ ev^* : H^*(K_j; \mathbf{Z}/p) \to H^*(S^1 \times \Omega K_j; \mathbf{Z}/p) \to H^{*-1}(\Omega K_j; \mathbf{Z}/p).$$

Here ΩX means the space of continuous loops on X which map $1 \in S^1$ to the base point of X. Let $\sigma^* : H^*(K_j; \mathbf{Z}/p) \to H^{*-1}(\Omega K_j; \mathbf{Z}/p)$ be the cohomology suspension. Since $i^*(\bar{\iota}_j) = \sigma^*(\iota_j)$ and $\sigma^*(\iota_j)$ is the fundamental element in $H^{n_j-1}(\Omega K_j; \mathbf{Z}/p) = H^{n_j-1}(K(\mathbf{Z}/p, n_j - 1); \mathbf{Z}/p)$, we see that if $\tilde{\mathcal{D}}(\iota_j) = \sigma^*(\iota_j)$ then $\mathcal{D}_{K_j,p}(\iota_j) = \bar{\iota}_j$. We now prove $\tilde{\mathcal{D}}(\iota_j) = \sigma^*(\iota_j)$.

Let $f : (I^{n_j}, \partial I^{n_j}) \to (K_j, *)$ be a continuous map and $g : I^{n_j-1} \to \Omega K_j$ a map defined by $g(t)(s) = f(t, s)$ for $t \in I^{n_j-1}$ and $s \in I$. The argument of the proof of [**36**, Proposition 2.1] enables us to deduce that

$$\tilde{\mathcal{D}}(dual[f]) = dual[g]$$

for the element $dual[f]$ in $dual(\pi_{n_j}(K_j) \otimes \mathbf{Z}/p)$ and $dual[g]$ in $dual(\pi_{n_j-1}(\Omega K_j) \otimes \mathbf{Z}/p)$ under the isomorphisms $H^{n_j}(K_j; \mathbf{Z}/p) \cong dual(H_{n_j}(K_j; \mathbf{Z}/p)) \cong dual(\pi_{n_j}(K_j) \otimes \mathbf{Z}/p)$ and $H^{n_j-1}(\Omega K_j; \mathbf{Z}/p) \cong dual(\pi_{n_j-1}(\Omega K_j) \otimes \mathbf{Z}/p)$.

On the other hand, let $\varepsilon_1 : \Omega K_j \to PK_j \to K_j$ be the path-loop fibration defined by $\varepsilon_1(\gamma) = \gamma(1)$ for $\gamma \in PK_j$. Here $PK_j = \{\gamma \in Map([0,1], K_j); \gamma(0) = *\}$ Then there is the following commutative diagram

$$\begin{array}{ccccc}
\pi_{n_j-1}(\Omega K_j) & \xleftarrow{\partial} & \pi_{n_j}(PK_j,\Omega K_j) & \xrightarrow{\varepsilon_{1*}} & \pi_{n_j}(K_j) \\
\downarrow h & & \downarrow h & & \downarrow h \\
H_{n_j-1}(\Omega K_j;\mathbf{Z}) & \xleftarrow{\partial} & H_{n_j}(PK_j,\Omega K_j;\mathbf{Z}) & \xrightarrow{\varepsilon_{1*}} & H_{n_j}(K_j;\mathbf{Z}),
\end{array}$$

where ∂ are connecting homomorphisms h are Hurewicz isomorphisms. We define the map

$$\tilde{f} : (I^{n_j}, \partial I^{n_j}, I^{n_j-1}\times 0 \cup \partial I^{n_j-1}\times I) \to (PK_j, \Omega K_j, *)$$

by

$$\tilde{f}(t,s)(u) = * \text{ if } 0 \le u \le 1/(s+1) \quad \text{and}$$

$$\tilde{f}(t,s)(u) = f(t, u(s+1)-1) \text{ if } 1/(s+1) \le u \le 1.$$

Then we can deduce that $\varepsilon_{1*}(\tilde{f}) = f$ and that $\partial(\tilde{f})$ is homotopic to g by the homotopy

$$H : (I^{n_j-1}\times I, \partial I^{n_j-1}\times I) \to (\Omega K_j, *)$$

defined by

$$H(t,l)(u) = * \text{ if } 0 \le u \le l/2 \quad \text{and}$$

$$H(t,l)(u) = f(t, (2u-l)/(2-l)) \text{ if } l/2 \le u \le 1.$$

Thus it follows that $\varepsilon_{1*}\partial^{-1}([g]) = [f]$, that is, for the cohomology suspension map σ^*,

$$\sigma^*(dual[f]) = dual[g].$$

This completes the proof. □

We can determine the structure of the kernel of the D-map $\mathcal{D}_{X,p} : H^4(X;\mathbf{Z}/p) \to H^3(LX;\mathbf{Z}/p)$ completely. Let $\eta_p : H^4(X;\mathbf{Z}) \to H^4(X;\mathbf{Z}/p)$ be the mod p reduction. Then we have

PROPOSITION 3.3. *Suppose that X is a simply connected space and that there exists a morphism of algebras ψ to $H^*(X;\mathbf{Z}/p)$ from an algebra $\Lambda(y_1,\dots,y_l)\otimes \mathbf{Z}/p[x_1,\dots,x_n]/(\rho_1,\dots,\rho_m)$ which is an isomorphism below degree 4, where $\deg\rho_i = 4$. If x is an element in the kernel of the D-map $\mathcal{D}_X = \int_{S^1}\circ ev^* : H^4(X;\mathbf{Z}) \to H^3(LX;\mathbf{Z})$, then $\eta_2(x) = \sum \lambda_i x_i^2$ for some constant λ_i and $\eta_p(x) = 0$ if $p \neq 2$.*

PROOF. By virtue of Theorem 2.4 and Theorem 3.1, we see that $\mathcal{D}_{X,p}$ coincides with the operator $\sum_{i=1}^n \frac{\partial}{\partial x_i}\bar{x}_i$. Therefore, the image of the map $\mathcal{D}_{X,p}$ is included in the image of ψ of Theorem 2.4. Hence, we can deduce that if $\mathcal{D}_{X,p}(z) = 0$ for some $z \in H^4(X;\mathbf{Z}/p)$, then

$$\sum_{i=1}^n \frac{\partial z}{\partial x_i}\bar{x}_i \in (d(\omega_1),..,d(\omega_m))_{A_p}.$$

We can write $z = z_1 + z_2$ by using elements z_1 and z_2 which are linear combinations of x_ix_j and x_k respectively. From the definition of $d(\omega_j)$, it follows that $z_2 = 0$ and

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}\bar{x}_i(z_1 - \sum \xi_j\rho_j) = 0$$

in A_p for some ξ_j in $\mathbf{Z}/p$. Since $\deg\rho_i = 4$ and $\deg \frac{\partial}{\partial x_i}(z_1 - \sum\xi_j\rho_j) = 2$ for any element x_i with degree 2, one can conclude that $\frac{\partial}{\partial x_i}(z_1 - \sum\xi_j\rho_j) = 0$ in $\mathbf{Z}/p[x_1,..,x_n]$ for any i. Thus, in $\mathbf{Z}/p[x_1,..,x_n]/(\rho_1,..,\rho_m)$, $z_1 = 0$ if $p \neq 2$ and $t_1 = \sum \xi_i x_i^2$ if $p = 2$. □

THEOREM 3.4. *Let X be a simply connected space. Suppose that*
(i): $H^4(X;\mathbf{Z}) \cong \mathbf{Z}\oplus\cdots\oplus\mathbf{Z}\oplus\mathbf{Z}/p_1\oplus\cdots\oplus\mathbf{Z}/p_k$, where p_i is prime for any i, and
(ii): $x^2 = 0$ for any element $x \in H^2(X;\mathbf{Z}/2)$ if $H^4(X;\mathbf{Z})$ has 2-torsion.
Then the D-map of X is good.

Theorem 3.4 is a generalization of Theorems 1.8 and ??.

PROOF OF THEOREM 3.4. For any element x in $\mathrm{Ker}\{\mathcal{D}_X : H^4(X;\mathbf{Z}) \to H^3(LX;\mathbf{Z})\}$, its mod p reduction $\eta_p(x)$ is zero if $p \neq 2$ by Proposition 3.3. Therefore it follows from (i) that the free part and odd torsion part of x is zero. Moreover, the condition (ii) enables us to deduce that the 2-torsion part of x is zero. □

4. Homogeneous spaces of rank one

We recall that the exceptional Lie group G_2 is given by $\{x \in Iso(\mathcal{C}, \mathcal{C}); x(uv) = x(u)x(v)$ for any $u, v \in \mathcal{C}\}$, where $\mathcal{C}$ is the Cayley algebra and $Iso(\mathcal{C}, \mathcal{C})$ is the group of all $\mathbf{R}$-isomorphisms from $\mathcal{C}$ to $\mathcal{C}$ itself and the group structure is given by the composition of maps and that the special unitary group $SU(n)$ is given by $\{A \in M(n, \mathbf{C}); A^*A = I, \det A = 1\}$, where A^* means the transposed conjugate matrix of A and I means the identity matrix. It is known the following fundamental theorems;

THEOREM 4.1 ([**47**, Künneth theorem]). *Let X and Y be spaces and $H_i(X; \mathbf{Z})$ is finitely generated for any i. Then there is a short exact sequence:*

$$0 \to \oplus_{p+q=n} H^p(X; \mathbf{Z}) \otimes H^q(Y; \mathbf{Z}) \to H^n(X \times Y; \mathbf{Z}) \to \oplus_{p+q=n+1} Tor(H^p(X; \mathbf{Z}), H^q(Y; \mathbf{Z})) \to 0.$$

THEOREM 4.2 ([**47**, Universal-coefficient theorem]). *For any module G, there is a short exact sequence:*

$$0 \to Ext(H_{n-1}(X; \mathbf{Z}), G) \to H^n(X; G) \to Hom(H_n(X; \mathbf{Z}), G) \to 0$$

In general, let $F \to E \to B$ be a fibration in which the base B is a connected CW complex with $\pi_1(B)$ acting trivially on $H^*(F; G)$, where G is an abelian group. Recall

THEOREM 4.3 ([**41**, Leray-Serre theorem]). *There is a first quadrant spectral sequence $E^r_{p,q}$ such that $E^1_{p,q} \cong C_p(B; H_q(F; G))$ and $E^2_{p,q} \cong H_p(B; H_q(F; G))$ which converges to $H_*(E; G)$.*

In this section, $\{E^r_{*,*}, d^r\}$ means a homology Leray-Serre spectral sequence.

The classification of compact, simply connected, homogeneous space of rank one has been made by Oniscik [**37**]. In [**35**], McCleary and Ziller have determined the mod p cohomology of the homogeneous spaces completely for any prime p. These results are also used to prove Theorem 1.2. According to Table [**35**, p.767], we now list such homogeneous spaces which are not diffeomorphic to spheres or projective spaces:

(1) $(SO(2n+1), SO(2n-1) \times SO(2), 1)$, $(n \geq 2)$
(2) $(SO(2n+1), SO(2n-1), 1)$, $(n \geq 2)$
(3) $(SU(3), SO(3), 4)$,
(4) $(Sp(2), SU(2), 10)$,
(5) $(G_2, SO(4), (1,3))$,
(6) $(G_2, U(2), 3)$,
(7) $(G_2, SU(2), 3)$,
(8) $(G_2, SO(3), 4)$,
(9) $(G_2, SO(3), 28)$,

where (1),(2) and (3) are standard inclusions. (4),(5) and (9) are maximal inclusion. (6) is by $U(2) \subset SO(4)$. (7) is by $SU(2) \subset U(3) \subset SO(4)$. (8) is by $SO(3) \subset SO(4)$. Here the triple (G, H, i) consisting of the Lie group G, the subgroup H and the integer or a pair of integers means the homogeneous space G/H of G by the subgroup H with the index i. Here the index of the subgroup H of G is that of the subalgebra $Lie(H)$ of the Lie algebra $Lie(G)$ of G in the sense of Dynkin ([**10**]). If a Lie group H has k simple factors, then $\pi_3(H)$ is isomorphic to a free abelian group of rank k, that is, $\pi_3(H) = \oplus^k \mathbf{Z}$. It is known that $\pi_3(H) = \mathbf{Z}$ if $H \neq SO(4)$ and $\pi_3(H) = \mathbf{Z} \oplus \mathbf{Z}$ if $H = SO(4)$ ([**32**]). Therefore in the above cases we can regard $j_* : \pi_3(H) \to \pi_3(G)$ as the multiplication by an integer n or a pair of integers (m, n) associate to the inclusion $j : H \to G$ if $H \neq SO(4)$ or $H = SO(4)$, respectively. The assertion of [**37**, Lemma 4] guarantees that the index of the subgroup H of G can be interpreted as the above integer or pair of integers determined by the inclusion j.

PROPOSITION 4.4. *Any compact, simply connected homogeneous space M of rank one satisfies the conditions (i) and (ii) in Theorem 3.4.*

PROOF. It is clear that spheres and projective spaces satisfy (i) and (ii). We will show that the nine homogeneous spaces listed above satisfy (i) and (ii). From the computation of the cohomology of the homogeneous spaces M by McCleary and Ziller [**35**, Theorem 1], the cohomology algebra, $H^*(M) = H^*(M; \mathbf{Z}/p)$, is given by the table as follows:

$$
\begin{aligned}
(1):\ H^*(M) &\cong H^*(P^{2n-1}\mathbf{C}) \text{ if } p \neq 2 \\
&\cong H^*(S^{2n} \times P^{n-1}\mathbf{C}) \text{ if } p = 2 \\
(2):\ H^*(M) &\cong H^*(S^{4n-1}) \text{ if } p \neq 2 \\
&\cong H^*(S^{2n-1} \times S^{2n}) \text{ if } p = 2 \\
(3):\ H^*(M) &\cong H^*(S^5) \text{ if } p \neq 2 \\
&\cong H^*(S^2 \times S^3) \text{ if } p = 2 \\
(4):\ H^*(M) &\cong H^*(S^7) \text{ if } p \neq 2 \text{ or } p \neq 5 \\
&\cong H^*(S^3 \times S^4) \text{ if } p = 2 \text{ or } p = 5 \\
(5):\ H^*(M) &\cong H^*(P^2\mathbf{H}) \text{ if } p \neq 2 \\
&\cong \mathbf{Z}/2[x, y]/(x^3 = y^2, x^2y = 0) \text{ if } p = 2 \text{ where } \deg x = 2 \text{ and } \deg y = 3 \\
(6):\ H^*(M) &\cong H^*(P^5\mathbf{C}) \text{ if } p \neq 2 \text{ or } p \neq 3 \\
&\cong H^*(S^2 \times P^2\mathbf{H}) \text{ if } p = 3 \\
&\cong H^*(S^6 \times P^2\mathbf{C}) \text{ if } p = 2 \\
(7):\ H^*(M) &\cong H^*(S^{11}) \text{ if } p \neq 2 \text{ or } p \neq 3 \\
&\cong H^*(S^3 \times P^2\mathbf{H}) \text{ if } p = 3 \\
&\cong H^*(S^5 \times S^6) \text{ if } p = 2 \\
(8):\ H^*(M) &\cong H^*(S^{11}) \text{ if } p \neq 2 \\
&\cong H^*(S^2 \times S^3 \times S^6) \text{ if } p = 2 \\
(9):\ H^*(M) &\cong H^*(S^{11}) \text{ if } p \neq 2 \text{ or } p \neq 7 \\
&\cong H^*(S^3 \times P^2\mathbf{H}) \text{ if } p = 7 \\
&\cong H^*(S^2 \times S^3 \times S^6) \text{ if } p = 2,
\end{aligned}
$$

as algebras. Here $P^n\mathbf{C}$ and $P^n\mathbf{H}$ mean the complex and quaternion projective space, respectively. Then one can conclude that $H^4(M; \mathbf{Z})$ is torsion free for the cases (1), (3), (5), (6) and (8).

We consider the case (9). Let $\pi : Spin(n) \to SO(n)$ be the universal covering. By the Hurewicz theorem, $(j\pi)_* : H_3(Spin(3); \mathbf{Z}) \to H_3(G_2; \mathbf{Z})$ is multiplication by 28. In order to prove that $j_* : H_3(SO(3); \mathbf{Z}) \to H_3(G_2; \mathbf{Z})$ is multiplication by 14, we will show that $\pi_* : H_3(Spin(3); \mathbf{Z}) \to H_3(SO(3); \mathbf{Z})$ is multiplication by 2.

Let us consider the homology Leray-Serre spectral sequence $\{E^r_{*,*}, d^r\}$ of the universal $SO(n)$-bundle. Since $E^2_{2,1} = H_2(BSO(3); H_1(SO(3); \mathbf{Z})) = H_2(BSO(3); \mathbf{Z}/2) = \mathbf{Z}/2$ and $E^2_{4,0} = H_4(BSO(3); \mathbf{Z}) = \mathbf{Z}$, it follows that $E^3_{4,0} = 2\mathbf{Z}$. Therefore we can deduce $d^3 : E^3_{4,0} \to E^3_{0,3}$ is multiplication by $1/2$. Note that $E^3_{0,3} = H_3(SO(3); \mathbf{Z}) = \mathbf{Z}$. The index of the map $B(\pi)^* : H^4(BSO(3); \mathbf{Z}) = \mathbf{Z} \to H^4(BSpin(3); \mathbf{Z}) = \mathbf{Z}$ is 4. From the universal coefficient theorem, it follows that the index of the map $B(\pi)_* : H_4(BSpin(3); \mathbf{Z}) \to H_4(BSO(3); \mathbf{Z})$ is 4 also. Thus the naturality of the differential in the spectral sequence enables us to conclude that the index of $\pi_* : H_3(Spin(3); \mathbf{Z}) \to H_3(SO(3); \mathbf{Z})$ is 2. Hence we see $j_* : H_3(SO(3); \mathbf{Z}) \to H_3(G_2; \mathbf{Z})$ is multiplication by 14.

To prove that the homogeneous space M of the case (9) satisfies the condition (i), we consider the homology Leray-Serre spectral sequence $\{E^r_{*,*}, d^r\}$ of the fibration $SO(3) \to G_2 \to G_2/SO(3) = M$. Let $\{F_pH_*\}_{p \geq 0}$ be the filtration of $H_*(G_2; \mathbf{Z})$ which comes from the spectral sequence $\{E^r_{*,*}, d^r\}$. Notice that $j_* : H_3(SO(3); \mathbf{Z}) \to H_3(G_2; \mathbf{Z})$ coincides with the boundary homomorphism

$$H_3(SO(3); \mathbf{Z}) = E^2_{0,3} \to E^\infty_{0,3} \cong E^0_{0,3} = F_0H_3 \subset F_1H_3 \subset F_2H_3 \subset F_3H_3 = H_3(G_2; \mathbf{Z}).$$

Since $E^2_{0,2} = H_2(SO(3); \mathbf{Z}) = 0$ and $E^2_{1,1} = 0$, we obtain $H_3(M; \mathbf{Z}) = E^2_{3,0} = E^\infty_{3,0} \cong F_3H_3/F_2H_3$. From the table above, it follows that $E^2_{2,1} = \mathbf{Z}/2$ and $H_4(M; \mathbf{Z})$ does not have 2-torsion part and a free part. Therefore we see $\mathbf{Z}/2 = E^2_{2,1} \cong E^\infty_{2,1} \cong F_2H_3/F_1H_3$. The fact that the index of j is non-zero and $E^0_{0,3}$ is a subgroup of $H_3(G_2; \mathbf{Z}) = \mathbf{Z}$ allows us to deduce $E^2_{0,3} \cong E^\infty_{0,3}$. Since M is simply connected, clearly $E^2_{1,2} = 0$. Hence j_* coincides with the inclusion $\mathbf{Z} = H_3(SO(3); \mathbf{Z}) = E^2_{0,3} \cong E^\infty_{0,3} \cong E^0_{0,3} = F_0H_3 = F_1H_3 \subset F_2H_3 \subset F_3H_3 = H_3(G_2; \mathbf{Z})$. The above argument yields that the inclusion $\mathbf{Z} = F_1H_3 \to F_2H_3 = \mathbf{Z}$ is multiplication by 2. Since the index of j_* is 14, we have the inclusion $\mathbf{Z} = F_2H_3 \to F_3H_3 = \mathbf{Z}$ is multiplication by 7. It turns out that $H_3(M;\mathbf{Z}) = \mathbf{Z}/7$. By using the universal coefficient theorem, the torsion part of $H^4(M;\mathbf{Z})$ is $\mathbf{Z}/7$. Hence we see the manifold $M = G_2/SO(3)$ satisfies the condition (i). The same argument works in cases (2), (4) and (7).

From the table of above ([**35**, Theorem 1]), we obtain that the case where $H^4(M;\mathbf{Z})$ has 2-torsion is (2) for $n = 2$ and (4). It is clear that the manifold satisfies the condition (ii) since we can see that the manifolds $SO(5)/SO(3)$ and $Sp(2)/SU(2)$ are 2-connected by homotopy exact sequences. □

THEOREM 4.5. *Let M be a simply connected 4-dimensional manifold, compact simply connected homogeneous spaces of rank one or a finite product of those manifolds. Then the D-map of M is good. Therefore the string class $\mu(Q)$ vanishes if and only if $\frac{1}{2}p_1(\xi)$ vanishes.*

PROOF. Let X and Y be simply connected spaces satisfying the condition (i) in Theorem 3.4. By the Universal Coefficient Theorem, we see that $H^2(X;\mathbf{Z})$ and $H^2(Y;\mathbf{Z})$ are torsion free. Hence it follows from the Künneth Theorem that $H^4(X \times Y;\mathbf{Z})$ is isomorphic to $\oplus_{i+j=4} H^i(X;\mathbf{Z}) \otimes H^j(Y;\mathbf{Z})$. As a consequence, the product space $X \times Y$ also satisfies the condition (i) in Theorem 3.4. It is clear that if X and Y satisfy the condition (ii) in Theorem 3.4, then $X \times Y$ also satisfy the condition (ii) in Theorem 3.4. Let M be a simply connected 4-dimensional manifold. Since $H^4(M;\mathbf{Z})$ is isomorphic to $\mathbf{Z}$, it follows that M satisfies the condition (i) in Theorem 3.4. Therefore this theorem follows from Proposition 4.4. □

REMARK 4.6. *For the manifolds M in the cases (1), (3), (5), (6) and (8), $H^4(M;\mathbf{Z})$ is torsion free. Therefore, by virtue of Theorem 1.8, we can deduce that the D-maps of these manifolds are good. Since the manifolds in the case (2), (4) and (7) are 2-connected, it follows from* [**36**, Theorem 3.1] *that the manifolds have the good D-map. However, we cannot conclude that the manifold in the case (9) and product spaces of compact, simply connected homogeneous spaces of rank one have the good D-map by applying Theorem 1.8 or* [**36**, Theorem 3.1].

CHAPTER 3

Rational cyclic cohomology and formality

In the following, we suppose that X is a connected and simply connected space such that $H^*(X;\mathbf{Q})$ is finitely generated as a $\mathbf{Q}$-algebra and suppose that X is not rationally contractible.

1. Normality of $ES^1 \times_{S^1} LX$

Let $\mathcal{A}$ be a CDGA. Recall the definition of formality in Chapter 0. Let $I(S)$ denote the ideal in the algebra $\wedge Z$ generated by a basis of a subspace S in Z, i.e., $I(S) = (S)_{\wedge Z}$. When $\mathcal{A}$ is formal, we can choose a minimal model $\mathcal{M} = (\wedge Z, d)$ of $\mathcal{A}$ such that $Z = \mathrm{Ker}(d|_Z) \oplus \mathrm{Ker}\ (\psi|_Z)$ for a quasi-isomorphism $\psi : \mathcal{M} \to H^*(\mathcal{A})$. Therefore, following [**8**, Theorem (4.1)],

$$\begin{gathered}\mathcal{A} \textit{ is formal if and only if there is a complement } N \textit{ to } Ker(d|_Z), \\ \textit{such that any d-cocycle of } I(N) \textit{ is d-exact.}\end{gathered} \tag{1.1}$$

Here we state about the construction of a minimal model and some properties of it. In the following, we use the symbol $[w]$ for the element which is represented by a cocycle w in a cohomology. According to [**8**, p.251], [**17**, IX,D] or [**38**, p.172], the minimal model $\mathcal{M}$ of $\mathcal{A}$ and $\rho : \mathcal{M} \to \mathcal{A}$ satisfying the condition (3) of Definition 1.1 is inductively constructed as $\mathcal{M} = \cup_n \mathcal{M}(n)$ and $\rho = \cup_n \rho_n$ with $\rho_n : \mathcal{M}(n) \to \mathcal{A}$, where $\mathcal{M}(n)$ is minimal and generated by elements in degrees $\leq n$, ρ_n^* is an isomorphism in degrees $\leq n$, and ρ_n^* is an injection in degree $n+1$. Here $\mathcal{M}(n+1)$ is defined by an elementary extension ([**17**, IX,D]) $\mathcal{M}(n+1) = \mathcal{M}(n) \otimes_d \wedge Z^{n+1}$ where $Z^{n+1} = C^{n+1} \oplus K^{n+1}$,

$$C^{n+1} = \mathrm{Coker}(\rho^* : H^{n+1}(\mathcal{M}(n)) \to H^{n+1}(\mathcal{A})),$$

$$K^{n+1} = \mathrm{Ker}(\rho^* : H^{n+2}(\mathcal{M}(n)) \to H^{n+2}(\mathcal{A})),$$

$d|_{C^{n+1}} = 0$, and passing to cohomology $[d|_{K^{n+1}}]$ is an injection. In the following, we put $C = \oplus_n C^n$, and $K = \oplus_n K^n$. We recall, in Definition 1.1 of Chapter 0, that $dz_i \in \wedge Z_{<i}$ for

a well ordered index I and $i \in I$. Since $(\wedge Z_{<i}, d)(\wedge Z_{<i}, d|_{\wedge Z_{<i}})$ is also a CDGA and dz_i is also a cocycle in $(\wedge Z_{<i}, d)$, the homology class of dz_i, $[dz_i]$, is defined in $H_*(\wedge Z_{<i}, d)$. Put

$$\text{Ker } [d|_Z] := \{z_i \in Z;\ \ dz_i \text{ is cohomologus to zero in the complex } (\wedge Z_{<i}, d)\}.$$

Then we see Ker $[d|_Z] \supset Ker(d|_Z)$.

Definition 1.1 ([**38**, Definition 1.2]). *A minimal model $(\wedge Z, d)$ is called* normal *if* $\text{Ker}[d|_Z] = \text{Ker}(d|_Z)$.

For an element z of Z^n, $z \in \text{Ker}[d|_Z]$ means that $dz = du$ for an element u of $\wedge Z^{<n}$. So it is clear that $\text{Ker}[d|_Z] \supset \text{Ker}(d|_Z)$ in general. Further we shall say in this paper, when we put $Z^{\leq n} = \oplus_{i \leq n} Z^i$, that

Definition 1.2. *A minimal model $(\wedge Z, d)$ is (n)-normal if* $\text{Ker}[d|_{Z^{\leq n}}] = \text{Ker}(d|_{Z^{\leq n}})$.

If $(\wedge Z, d)$ is (n)-normal, then it is (i)-normal for $i \leq n$. Also if $(\wedge Z, d)$ is (n)-normal and $\text{Ker}[d|_{Z^{n+1}}] = \text{Ker}(d|_{Z^{n+1}})$, then it is $(n+1)$-normal.

It is known ([**38**, p.172]) that the minimal model constructed above is normal and ([**38**, Lemma 1.8]) that

$$\begin{aligned} &\textit{If a minimal model } \mathcal{M} = (\wedge Z, d) \textit{ is normal and formal,} \\ &\textit{then } H^*(\mathcal{M}) \textit{ is generated by } [\text{Ker}(d|_Z)] \textit{ as an algebra.} \end{aligned} \tag{1.2}$$

Under the construction above, we have $C = \text{Ker}(d|_Z)$ and $Z = C \oplus K$ for $\mathcal{M} = (\wedge Z, d)$. Then we see that the condition (1.1) for the formality of $\mathcal{A}$ is modified as

Lemma 1.3. *$\mathcal{A}$ is formal if and only if any d-cocycle of $I(K)$ is d-exact.*

Proof . The "if" part is clear from (1.1) since K is a complement to C, $Z = C \oplus K$. The "only if" part is proved since there exists a quasi-isomorphism $\rho : \mathcal{M} \to (H^*(\mathcal{M}), 0)$ such that $\rho|_C = [id_C]$, $\rho|_K = 0$ and $\rho|_{d(K)} = [0]$ from (1.2). If an element w of $I(K)$ is closed, then $[w] = \rho(w) = 0$, and w is d-exact in $\mathcal{M}$ since ρ^* is an isomorphism on cohomology. □

Let $C_1 = C \oplus \beta(C)$, $K_1 = K \oplus \beta(K)$, and $I_1(S)$ the ideal in the algebra $\wedge \overline{Z} \otimes \wedge Z$ generated by a basis of a subspace S of $\overline{Z} \oplus Z$. Then we have $\text{Ker}(\delta|_{\overline{Z} \oplus Z}) = C_1$ and $\overline{Z} \oplus Z = C_1 \oplus K_1$.

Lemma 1.4. *If $(\wedge \overline{Z} \otimes \wedge Z, \delta)$ is (n)-normal and formal, then any δ-cocycle of $I_1({K_1}^{\leq n})$ is δ-exact.*

Proof. There exists a quasi-isomorphism $\rho : (\wedge \overline{Z} \otimes \wedge Z, \delta) \to (H^*(\wedge \overline{Z} \otimes \wedge Z, \delta), 0)$ such that $\rho|_{C_1} = [id_{C_1}]$ since $(\wedge \overline{Z} \otimes \wedge Z, \delta)$ is formal. Also since $(\wedge \overline{Z} \otimes \wedge Z, \delta)$ is (n)-normal, we have the following commutative diagram of CDGA-morphisms from Definition 1.2:

$$\begin{array}{ccc} (\wedge(\overline{Z} \oplus Z)^{\leq n}, \delta) & \xrightarrow{\ \pi\ } & (\wedge {C_1}^{\leq n}/(\delta(I_1({K_1}^{\leq n})) \cap \wedge {C_1}^{\leq n}), 0) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \phi} \\ (\wedge \overline{Z} \otimes \wedge Z, \delta) & \xrightarrow[\ \rho\ ]{} & (H^*(\wedge \overline{Z} \otimes \wedge Z, \delta), 0), \end{array}$$

where π is the projection with $\pi|_{{C_1}^{\leq n}} = [id_{{C_1}^{\leq n}}]$, $\pi|_{{K_1}^{\leq n}} = 0$ and $\pi|_{d({K_1}^{\leq n})} = [0]$, ϕ is defined by $[\phi|_{{C_1}^{\leq n}}] = [id_{{C_1}^{\leq n}}]$, and i is the inclusion induced by $(\overline{Z} \oplus Z)^{\leq n} \hookrightarrow \overline{Z} \oplus Z$. Then we have $\rho|_{{K_1}^{\leq n}} = \rho \circ i|_{{K_1}^{\leq n}} = \phi \circ \pi|_{{K_1}^{\leq n}} = 0$. If an element w of $I_1({K_1}^{\leq n})$ is a δ-cocycle, then $[w] = \rho(w) = 0$, and w is δ-exact in $(\wedge \overline{Z} \otimes \wedge Z, \delta)$ since ρ^* is an isomorphism on cohomology. □

Lemma 1.5. *If $(\wedge \overline{Z} \otimes \wedge Z, \delta)$ is formal, then it is normal.*

Proof. We show this lemma inductively. It is clear that $(\wedge \overline{Z} \otimes \wedge Z, \delta)$ is (1)-normal since

$$\text{Ker}[\delta|_{(\overline{Z} \oplus Z)^1}] = \text{Ker}[\delta|_{\overline{Z}^1}] = \overline{Z}^1 = \text{Ker}(\delta|_{\overline{Z}^1}) = \text{Ker}(\delta|_{(\overline{Z} \oplus Z)^1}).$$

Suppose that $(\wedge \overline{Z} \otimes \wedge Z, \delta)$ is (n)-normal. If we can show that $\text{Ker}[\delta|_{\overline{Z}^{n+1}}] = \beta(\text{Ker}[d|_{Z^{n+2}}])$, then we see that

$$\begin{aligned} \text{Ker}[\delta|_{(\overline{Z} \oplus Z)^{n+1}}] &= \text{Ker}[\delta|_{\overline{Z}^{n+1}}] \oplus \text{Ker}[d|_{Z^{n+1}}] \\ &= \beta(\text{Ker}[d|_{Z^{n+2}}]) \oplus \text{Ker}[d|_{Z^{n+1}}] \\ &= \beta(\text{Ker}(d|_{Z^{n+2}})) \oplus \text{Ker}(d|_{Z^{n+1}}) \\ &= \text{Ker}(\delta|_{\overline{Z}^{n+1}}) \oplus \text{Ker}(d|_{Z^{n+1}}) \\ &= \text{Ker}(\delta|_{(\overline{Z} \oplus Z)^{n+1}}), \end{aligned}$$

that is, $(\wedge\overline{Z}\otimes\wedge Z,\delta)$ is $(n+1)$-normal. The inclusion $\mathrm{Ker}[\delta|_{\overline{Z}^{n+1}}]\supset\beta(\mathrm{Ker}[d|_{Z^{n+2}}])$ is clear since $\delta(\overline{z})=\delta(\beta(u))$ if $d(z)=d(u)$ for elements z of Z^{n+2} and u of $(\wedge Z)^{n+2}$. Therefore we show that $\mathrm{Ker}[\delta|_{\overline{Z}^{n+1}}]\subset\beta(\mathrm{Ker}[d|_{Z^{n+2}}])$.

Suppose that $\overline{z}\in\mathrm{Ker}[\delta|_{\overline{Z}^{n+1}}]$, that is, there is an element w_1 in $(\wedge^1\overline{Z}\otimes\wedge Z)^{n+1}\cap I_1(K_1)$ such that

$$\delta(\overline{z})=\delta(w_1)$$

for an element $\overline{z}$ of $\overline{Z}^{n+1}$. If $\beta(w_1)=0$, we have $d(z)=d(u)$ for an element u of $(\wedge Z)^{n+2}$ such that $\beta(u)=w_1$, from (2.1). This means that $z\in\mathrm{Ker}[d|_{Z^{n+2}}]$. Let $\beta(w_1)\neq 0$. Then we have $\beta(w_1)\in I_2(K_1^{\leq n})$ and $\delta(\beta(w_1))=-\beta(\delta(w_1))=-\beta(\delta(\overline{z}))=\beta(\beta(d(z)))=0$. From Lemma 1.7, there is an element $w_2\in\wedge^2\overline{Z}\otimes\wedge Z$, such that

$$\delta(w_2)=\beta(w_1).$$

Then we have $\beta(w_2)\in I_1(K_1{}^{\leq n})$ and $\delta(\beta(w_2))=\delta(\beta(w_2))=-\beta(\delta(w_2))=-\beta(\beta(w_1))=0$. From Lemma 1.7, there is an element $w_3\in\wedge^3\overline{Z}\otimes\wedge Z$, such that

$$\delta(w_3)=\beta(w_2).$$

Iterating this argument yields an element $w_m\in\wedge^m\overline{Z}\otimes\wedge Z$, such that

$$\delta(w_m)=\beta(w_{m-1})\text{ and }\beta(w_m)=0.$$

From (2.1), there is an element u_{m-1} in $\wedge^{m-1}\overline{Z}\otimes\wedge Z$ such that

$$\beta(u_{m-1})=w_m$$

and there is an element u_{m-2} in $\wedge^{m-2}\overline{Z}\otimes\wedge Z$ such that

$$\beta(u_{m-2})=\delta(u_{m-1})+w_{m-1}$$

since $\beta(\delta(u_{m-1})+w_{m-1})=-\delta(\beta(u_{m-1}))+\beta(w_{m-1})=-\delta(w_m)+\beta(w_{m-1})=-\beta(w_{m-1})+\beta(w_{m-1})=0$.

Iterating this argument for $i=m-3,..,0$, there is an element u_0 in $\wedge Z$ such that

$$\beta(u_0)=\delta(u_1)+w_1.$$

Then $\beta(d(u_0))=-\delta(\beta(u_0))=-\delta(w_1)=-\delta(\overline{z})=\beta(d(z))$. Thus we have $d(z)=d(u_0)$ from (2.1), that is, $z\in\mathrm{Ker}[d|_{Z^{n+2}}]$. Hence $\mathrm{Ker}[\delta|_{\overline{Z}^{n+1}}]\subset\beta(\mathrm{Ker}[d|_{Z^{n+2}}])$. □

Lemma 1.6. *LX is formal if and only if any δ-cocycle of $I_1(K_1)$ is δ-exact.*

Proof. The "if" part follows from (1.1) since C_1 is the subspace of δ-cocycles in $\overline{Z}\oplus Z$ and K_1 is a complement to C_1: $\overline{Z}\oplus Z=C_1\oplus K_1$.

The "only if" part follows as in the proof of Lemma 1.3 since $(\wedge\overline{Z}\otimes\wedge Z,\delta)$ is normal from Lemma 1.5. □

Recently, N.Dupont and M.Vigué-Poirrier showed that when $H^*(X;\mathbf{Q})$ is finitely generated, LX is formal if and only if $H^*(X;\mathbf{Q})$ is free, i.e., X has the rational homotopy type of a product of Eilenberg Maclane spaces ([**9**]). As an equivariant version, we consider the necessary and sufficient condition for the formality of $ES^1\times_{S^1}LX$. For this purpose, we give a equivariant version of Lemma 1.6 in this section.

Let $C_2=\mathbf{Q}\{t\}\oplus\beta(C)$, $K_2=C\oplus K_1$, and $I_2(S)$ the ideal in the algebra $\mathbf{Q}[t]\otimes\wedge\overline{Z}\otimes\wedge Z$ generated by a basis of a subspace S of $\mathbf{Q}\{t\}\oplus\overline{Z}\oplus Z$. Then we have $\mathrm{Ker}(D|_{\mathbf{Q}\{t\}\oplus\overline{Z}\oplus Z})=C_2$ and $\mathbf{Q}\{t\}\oplus\overline{Z}\oplus Z=C_2\oplus K_2$.

Lemma 1.7. *If $(\mathbf{Q}[t]\otimes\wedge\overline{Z}\otimes\wedge Z,D)$ is (n)-normal and formal, then any D-cocycle of $I_2(K_2{}^{\leq n})$ is D-exact.*

Proof. There exists a quasi-isomorphism $\rho:(\mathbf{Q}[t]\otimes\wedge\overline{Z}\otimes\wedge Z,D)\to(H^*(\mathbf{Q}[t]\otimes\wedge\overline{Z}\otimes\wedge Z,D),0)$ such that $\rho|_{C_2}=[id_{C_2}]$ since $(\mathbf{Q}[t]\otimes\wedge\overline{Z}\otimes\wedge Z,D)$ is formal. Also since $(\mathbf{Q}[t]\otimes\wedge\overline{Z}\otimes\wedge Z,D)$ is (n)-normal, we have the following commutative diagram of CDGA-morphisms from Definition 1.2:

$$\begin{array}{ccc}
(\wedge(\mathbf{Q}\{t\}\oplus\overline{Z}\oplus Z)^{\leq n},D) & \xrightarrow{\ \pi\ } & (\wedge C_2{}^{\leq n}/(D(I_2(K_2{}^{\leq n}))\cap\wedge C_2{}^{\leq n}),0)\\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \phi}\\
(\mathbf{Q}[t]\otimes\wedge\overline{Z}\otimes\wedge Z,D) & \xrightarrow[\ \rho\ ]{} & (H^*(\mathbf{Q}[t]\otimes\wedge\overline{Z}\otimes\wedge Z,D),0),
\end{array}$$

where π is the projection with $\pi|_{C_2{}^{\leq n}}=[id_{C_2{}^{\leq n}}]$, $\pi|_{K_2{}^{\leq n}}=0$ and $\pi|_{d(K_2{}^{\leq n})}=[0]$, ϕ is defined by $[\phi|_{C_2{}^{\leq n}}]=[id_{C_2{}^{\leq n}}]$, and i is the inclusion induced by $(\mathbf{Q}\{t\}\oplus\overline{Z}\oplus Z)^{\leq n}\hookrightarrow\mathbf{Q}\{t\}\oplus\overline{Z}\oplus Z$. Then we have $\rho|_{K_2{}^{\leq n}}=\rho\circ i|_{K_2{}^{\leq n}}=\phi\circ\pi|_{K_2{}^{\leq n}}=0$. If an element w of $I_2(K_2{}^{\leq n})$ is a

D-cocycle, then $[w] = \rho(w) = 0$, and w is D-exact in $(\mathbf{Q}[t] \otimes \wedge\overline{Z} \otimes \wedge Z, D)$ since ρ^* is an isomorphism on cohomology. □

LEMMA 1.8. *If $(\mathbf{Q}[t] \otimes \wedge\overline{Z} \otimes \wedge Z, D)$ is formal, then it is normal.*

PROOF. We show this lemma inductively. It is clear that $(\mathbf{Q}[t] \otimes \wedge\overline{Z} \otimes \wedge Z, D)$ is (1)-normal since

$$\mathrm{Ker}[D|_{(\mathbf{Q}\{t\}\oplus\overline{Z}\oplus Z)^1}] = \mathrm{Ker}[D|_{\overline{Z}^1}] = \overline{Z}^1 = \mathrm{Ker}(D|_{\overline{Z}^1}) = \mathrm{Ker}(D|_{(\mathbf{Q}\{t\}\oplus\overline{Z}\oplus Z)^1}).$$

Suppose that $(\mathbf{Q}[t] \otimes \wedge\overline{Z} \otimes \wedge Z, D)$ is (n)-normal. If we can show that $\mathrm{Ker}[\delta|_{\overline{Z}^{n+1}}] = \beta(\mathrm{Ker}[d|_{Z^{n+2}}])$, then we see that

$$\begin{aligned}
\mathrm{Ker}[D|_{(\mathbf{Q}\{t\}\oplus\overline{Z}\oplus Z)^{n+1}}] &= \mathbf{Q}\{\delta_n \cdot t\} \oplus \mathrm{Ker}[\delta|_{\overline{Z}^{n+1}}] \\
&= \mathbf{Q}\{\delta_n \cdot t\} \oplus \beta(\mathrm{Ker}[d|_{Z^{n+2}}]) \\
&= \mathbf{Q}\{\delta_n \cdot t\} \oplus \beta(\mathrm{Ker}(d|_{Z^{n+2}})) \\
&= \mathbf{Q}\{\delta_n \cdot t\} \oplus \mathrm{Ker}(\delta|_{\overline{Z}^{n+1}}) \\
&= \mathrm{Ker}(D|_{(\mathbf{Q}\{t\}\oplus\overline{Z}\oplus Z)^{n+1}}),
\end{aligned}$$

where δ_n is 1 if $n = 1$ and 0 if $n > 1$, that is, $(\mathbf{Q}[t] \otimes \wedge\overline{Z} \otimes \wedge Z, D)$ is $(n+1)$-normal. The inclusion $\mathrm{Ker}[\delta|_{\overline{Z}^{n+1}}] \supset \beta(\mathrm{Ker}[d|_{Z^{n+2}}])$ is clear since $\delta(\overline{z}) = \delta(\beta(u))$ if $d(z) = d(u)$ for elements z of Z^{n+2} and u of $(\wedge Z)^{n+2}$. Therefore we show that $\mathrm{Ker}[\delta|_{\overline{Z}^{n+1}}] \subset \beta(\mathrm{Ker}[d|_{Z^{n+2}}])$.

Suppose that $\overline{z} \in \mathrm{Ker}[\delta|_{\overline{Z}^{n+1}}]$, that is, there is an element w_1 in $(\wedge^1\overline{Z} \otimes \wedge Z)^{n+1} \cap I_1(K_1)$ such that

$$\delta(\overline{z}) = \delta(w_1)$$

for an element $\overline{z}$ of $\overline{Z}^{n+1}$. If $\beta(w_1) = 0$, we have $d(z) = d(u)$ for an element u of $(\wedge Z)^{n+2}$ such that $\beta(u) = w_1$, from (2.1). This means that $z \in \mathrm{Ker}[d|_{Z^{n+2}}]$. Let $\beta(w_1) \neq 0$. Then we have $\beta(w_1) \in I_2(K_2^{\leq n})$ and $D(\beta(w_1)) = \delta(\beta(w_1)) = -\beta(\delta(w_1)) = -\beta(\delta(\overline{z})) = \beta(\beta(d(z))) = 0$. From Lemma 1.7, there is an element $w_2 = \sum_{i\geq 2} t^{i-2} \cdot w_{2,i}$ in $\mathbf{Q}[t] \otimes \wedge^{\geq 2}\overline{Z} \otimes \wedge Z$, where $w_{2,i} \in \wedge^i\overline{Z} \otimes \wedge Z$, such that

$$D(w_2) = \beta(w_1), \text{ especially } \delta(w_{2,2}) = \beta(w_1).$$

Then we have $\beta(w_{2,2}) \in I_2(K_2{}^{\leq n})$ and $D(\beta(w_{2,2})) = \delta(\beta(w_{2,2})) = -\beta(\delta(w_{2,2})) = -\beta(\beta(w_1)) = 0$. From Lemma 1.7, there is an element $w_3 = \sum_{i\geq 3} t^{i-3} \cdot w_{3,i}$ in $\mathbf{Q}[t] \otimes \wedge^{\geq 3}\overline{Z} \otimes \wedge Z$, where $w_{3,i} \in \wedge^i\overline{Z} \otimes \wedge Z$, such that

$$D(w_3) = \beta(w_{2,2}), \text{ especially } \delta(w_{3,3}) = \beta(w_{2,2}).$$

Iterating this argument yields an element $w_m = \sum_{i\geq m} t^{i-m} \cdot w_{m,i}$ in $\mathbf{Q}[t] \otimes \wedge^{\geq m}\overline{Z} \otimes \wedge Z$, where $w_{m,i} \in \wedge^i\overline{Z} \otimes \wedge Z$, such that

$$D(w_m) = \beta(w_{m-1,m-1}), \text{ especially } \delta(w_{m,m}) = \beta(w_{m-1,m-1}) \text{ and } \beta(w_{m,m}) = 0.$$

From (2.1), there is an element u_{m-1} in $\wedge^{m-1}\overline{Z} \otimes \wedge Z$ such that

$$\beta(u_{m-1}) = w_{m,m}$$

and there is an element u_{m-2} in $\wedge^{m-2}\overline{Z} \otimes \wedge Z$ such that

$$\beta(u_{m-2}) = \delta(u_{m-1}) + w_{m-1,m-1}$$

since $\beta(\delta(u_{m-1}) + w_{m-1,m-1}) = -\delta(\beta(u_{m-1})) + \beta(w_{m-1,m-1}) = -\delta(w_{m,m}) + \beta(w_{m-1,m-1}) = -\beta(w_{m-1,m-1}) + \beta(w_{m-1,m-1}) = 0$. Iterating this argument for $i = m-3, .., 0$, there is an element u_0 in $\wedge Z$ such that

$$\beta(u_0) = \delta(u_1) + w_1.$$

Then $\beta(d(u_0)) = -\delta(\beta(u_0)) = -\delta(w_1) = -\delta(\overline{z}) = \beta(d(z))$. Thus we have $d(z) = d(u_0)$ from (2.1), that is, $z \in \mathrm{Ker}[d|_{Z^{n+2}}]$. Hence $\mathrm{Ker}[\delta|_{\overline{Z}^{n+1}}] \subset \beta(\mathrm{Ker}[d|_{Z^{n+2}}])$. □

LEMMA 1.9. *$ES^1 \times_{S^1} LX$ is formal if and only if any D-cocycle of $I_2(K_2)$ is D-exact.*

PROOF. The "if" part follows from (1.1) since C_2 is the subspace of D-cocycles in $\mathbf{Q}\{t\} \oplus \overline{Z} \oplus Z$ and K_2 is a complement to C_2: $\mathbf{Q}\{t\} \oplus \overline{Z} \oplus Z = C_2 \oplus K_2$.

The "only if" part follows as in the proof of Lemma 1.3 since $(\wedge t \otimes \wedge\overline{Z} \otimes \wedge Z, D)$ is normal from Lemma 1.8. □

2. Formality conditions of $ES^1 \times_{S^1} LX$

Let BS^1 be the classifying space of S^1. Then we know that $H^*(BS^1;\mathbf{Q}) \cong \mathbf{Q}[t]$; the polynomial algebra generated by an element t with $\deg t = 2$. Let us return to the arguments of minimal models in Chapter 0. We know $HC^*(X;\mathbf{Q}) \cong H^*(ES^1 \times_{S^1} LX;\mathbf{Q}) \cong H^*(\mathbf{Q}[t] \otimes \wedge\overline{Z} \otimes \wedge Z, D)$ from [**4**] and (2.4), respectively. Let

$$T^* = H^*(\mathbf{Q}[t] \otimes \wedge^+(\overline{Z} \oplus Z), D).$$

Then we can decompose $H^*(\mathbf{Q}[t] \otimes \wedge\overline{Z} \otimes \wedge Z, D)$ into a direct sum of graded $\mathbf{Q}[t]$-modules $\mathbf{Q}[t] \oplus T^*$ since $\mathrm{Im}\, D \subset \mathbf{Q}[t] \otimes \wedge^+(\overline{Z} \oplus Z)$, i.e., as graded $\mathbf{Q}[t]$-modules:

$$H^*(ES^1 \times_{S^1} LX;\mathbf{Q}) \cong \mathbf{Q}[t] \oplus T^*. \tag{2.1}$$

M.Vigué-Poirrier proves in Lemma 2 of [**51**] that

$$T^* \cong \frac{\beta(\wedge\overline{Z} \otimes \wedge Z) \cap \mathrm{Ker}\delta}{\delta(\beta(\wedge\overline{Z} \otimes \wedge Z))} \tag{2.2}$$

as graded $\mathbf{Q}$-modules.

By using Lemma 1.9, we characterize the rational homotopy type of X when $ES^1 \times_{S^1} LX$ is formal as follows:

THEOREM 2.1. *The following are equivalent:*

1. *$ES^1 \times_{S^1} LX$ is formal,*
2. *X has the rational homotopy type of an odd dimensional sphere, and*
3. *$T^* \cong \mathbf{Q}\{z_i\}_{i>0}$ as graded $\mathbf{Q}$-modules with $\deg z_i$ even and* $\deg z_i = i \deg z_1$ *for any i.*

PROOF OF THE EQUIVALENCE OF (1) AND (2) OF THEOREM 2.1. The "if" part is proved as follows: Suppose $X_{(0)} \simeq {S^{2n+1}}_{(0)}$ for a positive integer n. Then we have $\mathcal{M}(X) = (\wedge(x_{2n+1}), 0)$ with x_{2n+1} a generator of the algebra $H^*(S^{2n+1};\mathbf{Q})$. By (2.4), there is a CDGA-isomorphism:

$$\mathcal{M}(ES^1 \times_{S^1} LX) \cong (\wedge(t, \overline{x}_{2n+1}, x_{2n+1}), D), \tag{2.3}$$

where $D(\overline{x}_{2n+1}) = 0$ and $D(x_{2n+1}) = t\overline{x}_{2n+1}$. Then we have $I_2(K_2) = I_2(x_{2n+1})$. Since every non-zero element of $I_2(x_{2n+1})$ is not a D-cocycle, we see that $ES^1 \times_{S^1} (S^{2n+1})^{S^1}$ is formal by Lemma 1.9.

The "only if" part is proved by using Lemma 1.9 and the following lemma:

LEMMA 2.2. *If $ES^1 \times_{S^1} LX$ is formal, then*

1. *X is formal,*
2. *there are elements y_{ij} of K such that $d(y_{ij}) = x_i x_j$ for any i, j where $\{x_1, \cdots, x_l\}$ is a basis of C, and*
3. *$C^{even} = 0$.*

By (1) of Lemma 2.2, X is formal. Then we see that

$$H^*(X;\mathbf{Q}) \cong \wedge C/(d(I(K)) \cap \wedge C) \tag{2.4}$$

as algebras from (1.2) . By (2) of Lemma 2.2, we see that (2.4) is given as follows:

$$H^*(X;\mathbf{Q}) \cong \wedge(x_1, x_2, \cdots, x_l)/(x_i x_j\ ; 1 \le i < j \le l),$$

where $\{x_1, \cdots, x_l\}$ is a basis of C. By (3) of Lemma 2.2, each x_i is of odd degree. We put $\deg x_i = 2n_i + 1$ for $i = 1, \cdots, l$.

Let $l > 1$. Then we can see, from a formal consequence of $H^*(X;\mathbf{Q})$ (Section 3 of [**21**]), that the minimal model of it is given as follows:

$$\mathcal{M}(X) \cong (\wedge(x_1, x_2, \cdots, x_l, y_{12}, \cdots, y_{l-1\,l}, v_1, v_2, \cdots),\ d\ ),$$

where $\deg x_i = 2n_i + 1$ $(1 \le i \le l)$, $\deg y_{ij} = 2n_i + 2n_j + 1$ $(1 \le i < j \le l)$, $\deg v_1 = 4n_1 + 2n_2 + 1$, $\deg v_2 = 2n_1 + 4n_2 + 1$, and the differential d is given as

$$d(x_1) = 0, \cdots, d(x_l) = 0,\ d(y_{ij}) = x_i x_j,\ d(v_1) = y_{12}x_1,\ d(v_2) = y_{12}x_2$$

for some elements of low degrees. Define an element w of $I_2(K_2)$ as

$$w = 2\ \overline{v}_1\overline{x}_2 - 2\ \overline{v}_2\overline{x}_1 + {\overline{y}_{12}}^2.$$

The element w is a D-cocycle since

$$\begin{aligned} D(w) &= \delta(w) \\ &= 2\delta(\overline{v}_1)\overline{x}_2 - 2\delta(\overline{v}_2)\overline{x}_1 + \delta({\overline{y}_{12}}^2) \\ &= 2\ (-\overline{y}_{12}x_1 + y_{12}\overline{x}_1)\overline{x}_2 - 2\ (-\overline{y}_{12}x_2 + y_{12}\overline{x}_2)\overline{x}_1 + 2\ \overline{y}_{12}(-\overline{x}_1 x_2 + x_1\overline{x}_2) \\ &= 0. \end{aligned}$$

Since $\delta(\wedge\overline{Z}\otimes\wedge Z)\subset\wedge\overline{Z}\otimes\wedge^{+}Z$, we have Im $D=$ Im $(\delta+t\cdot\beta)\subset\wedge\overline{Z}\otimes\wedge^{+}Z\oplus I_2(t)$. Thus we see that

$$\textit{any element of } \wedge\overline{Z} \textit{ is neither } \delta\textit{-exact nor } D\textit{-exact.} \tag{2.5}$$

From (2.5), w is not D-exact. Thus if $l>1$, we see that $ES^1\times_{S^1}LX$ is not formal from Lemma 1.9. Hence we have $l=1$, that is, $X_{(0)}\simeq S^{2n+1}{}_{(0)}$ for a positive integer n. □

PROOF OF THE EQUIVALENCE OF (2) AND (3) OF THEOREM 2.1. The "only if" part is proved as follows. Suppose $X_{(0)}\simeq S^{2n+1}{}_{(0)}$. By (2.3), we have

$$H^*(ES^1\times_{S^1}LX)\cong \mathbf{Q}[t,\overline{x}_{2n+1}]/(t\cdot\overline{x}_{2n+1})\cong\mathbf{Q}[t]\oplus\mathbf{Q}\{\overline{x}^i_{2n+1};\ i>0\}.$$

Thus $T^*\cong\mathbf{Q}\{\overline{x}^i_{2n+1};\ i>0\}$.

The "if" part is proved as follows. Suppose $C^{even}\neq 0$. Then for a non zero element x of C^{even}, the element $\overline{x}$ is a generator of $H^*(ES^1\times_{S^1}LX)$ such that its degree is odd. So we have

$$C^{even}=0.$$

Also suppose $\dim_{\mathbf{Q}}C^{odd}\geq 2$. If a basis of C^{odd} is given as $\{x_1,\cdots,x_n\}$, then $\overline{x}_1,\cdots,\overline{x}_n$ are the generators of even degrees of T^*. Let deg $\overline{x}_1\leq$ deg $\overline{x}_2$. Even if deg $\overline{x}_2=a\cdot$ deg $\overline{x}_1$ for some positive integer a, it contradicts to (3). For there are two elements of same degrees $\overline{x}_1^a$, $\overline{x}_2$ in T^*. So we have

$$\dim_{\mathbf{Q}}C^{odd}=1.$$

Since the differential d is decomposable, we have

$$K=0.$$

Thus we see that $H^*(X;\mathbf{Q})\cong\wedge(x_{2n+1})$ with the deg$x_{2n+1}=2n+1$. This means that $X_{(0)}\simeq S^{2n+1}_{(0)}$. □

PROOF OF (1) OF LEMMA 2.2. Let an element w of $I(K)$ be a d-cocycle. Then we see that $\beta(w)$ is a D-cocycle of $I_2(K_2)$ since $D(\beta(w))=-\beta(d(w))=0$. From Lemma 1.9,

there is an element $u=\sum_{i=1}^{n}t^{i-1}u_i$ in $\wedge t\otimes\wedge\overline{Z}\otimes\wedge Z$ for some n, where $u_i\in\wedge^i\overline{Z}\otimes\wedge Z$, such that

$$D(u)=\delta(u_1)=\beta(w).$$

This means that

$$\beta(u_i)=-\delta(u_{i+1})\text{ for } i=1,..,n-1 \text{ and } \beta(u_n)=0.$$

From (2.1), there is an element v_{n-1} in $\wedge^{n-1}\overline{Z}\otimes\wedge Z$ such that

$$\beta(v_{n-1})=u_n.$$

Then we have $\beta(\delta(v_{n-1}))=-\delta(\beta(v_{n-1}))=-\delta(u_n)=\beta(u_{n-1})$, that is, $\beta(\delta(v_{n-1})-u_{n-1})=0$. From (2.1), there is an element v_{n-2} in $\wedge^{n-2}\overline{Z}\otimes\wedge Z$ such that

$$\beta(v_{n-2})=\delta(v_{n-1})-u_{n-1}.$$

Then we have $\beta(\delta(v_{n-2}))=-\delta(\beta(v_{n-2}))=-\delta(\delta(v_{n-1}))+\delta(u_{n-1})=\delta(u_{n-1})=-\beta(u_{n-2})$, that is, $\beta(\delta(v_{n-2})+u_{n-2})=0$. From (2.1), there is an element v_{n-3} in $\wedge^{n-3}\overline{Z}\otimes\wedge Z$ such that

$$\beta(v_{n-3})=\delta(v_{n-2})+u_{n-2}.$$

Iterating this argument for $i=n-4,..,0$, we see that there is an element $v_0\in\wedge Z$ such that

$$\beta(v_0)=\delta(v_1)+(-1)^{n+1}u_1.$$

Then we have $\beta(d(v_0))=-\delta(\beta(v_0))=(-1)^n\delta(\delta(v_1))+(-1)^n\delta(u_1)=(-1)^n\delta(u_1)=(-1)^n\beta(w)$. From (2.1), we have

$$d((-1)^nv_0)=(-1)^n\delta(v_0)=w.$$

Thus X is formal from Lemma 1.3. □

PROOF OF (2) OF LEMMA 2.2. For any i, j where $\{x_1,\cdots,x_m\}$ is a basis of C, we see that

$$\beta(x_ix_j)=\overline{x}_ix_j+(-1)^{deg(x_i)}x_i\overline{x}_j\in I_2(K_2).$$

This element $\beta(x_i x_j)$ is a D-cocycle. By the same argument of the proof of (1) of Lemma 2.2, there is an element u in $I(K)$ such that $d(u) = x_i x_j \in \wedge^2 Z$. From the decomposability of d, we see that u is not decomposable, but an element of K. □

If $ES^1 \times_{S^1} LX$ is formal, then we see

$$\textit{there is no element } y \textit{ of } K - \{0\} \textit{ such that } d(y) \in \wedge C^{even}. \tag{2.6}$$

In fact, (2.6) is showed as follows: Suppose that there are an element y in K and an element f in $\wedge C^{even}$ such that $d(y) = f$. If a basis of C^{even} is $\{x_{i_1}, \cdots, x_{i_n}\}$, we have

$$\overline{y}\ \overline{x}_{i_1} \cdots \overline{x}_{i_n} \in I_2(K_2).$$

This element $\overline{y}\ \overline{x}_{i_1} \cdots \overline{x}_{i_n}$ is a D-cocycle since

$$D(\overline{y}\ \overline{x}_{i_1} \cdots \overline{x}_{i_n}) = \delta(\overline{y})\overline{x}_{i_1} \cdots \overline{x}_{i_n} = -\beta(f)\overline{x}_{i_1} \cdots \overline{x}_{i_n} = 0.$$

By (2.5), we see that $\overline{y}\ \overline{x}_{i_1} \cdots \overline{x}_{i_n}$ is not D-exact. Then $ES^1 \times_{S^1} LX$ is not formal from Lemma 1.9. Thus we have (2.6).

Proof of (3) of Lemma 2.2. Suppose $C^{even} \neq 0$. For a generator x of C^{even},

$$\beta(x^2) = 2\overline{x}x \in I_2(K_2).$$

This element $\beta(x^2) = 2\overline{x}x$ is a D-cocycle and not D-exact. This contradicts Lemma 1.9. Thus we have $C^{even} = 0$.

In fact, if it is D-exact, then there is an element $\overline{y}$ in $\overline{Z}$ such that $\delta(\overline{y}) = D(\overline{y}) = \beta(x^2)$ from the decomposability of D. Then there is an element y in K such that $d(y) = -x^2$ from (2.1). But we can not have such an element from (2.6). □

3. Connes' periodicity map on cyclic cohomology

Let $\mathcal{A} = (A, d)$ be a $\mathbf{Q}$-CDGA. We define the normalized cyclic bar complex $(\mathbf{Q}[t] \otimes \mathrm{N}(\mathcal{A}), b + t \cdot B)$ of $\mathcal{A}$ and the cyclic homology of $\mathcal{A}$ following [**15**, §2] :

$$\mathrm{N}(\mathcal{A}) = \sum_{k=0}^{\infty} A \otimes \bar{A}^{\otimes k},$$

the derivation $b = b_0 + b_1$ and B have the properties such that

$$b_0([z_0, .., z_k]) = -\sum_{i=0}^{k} (-1)^{\varepsilon_{i-1}} [z_0, .., z_{i-1}, d(z_i), z_{i+1}, .., z_k],$$

$$b_1([z_0, .., z_k]) = -\sum_{i=0}^{k-1} (-1)^{\varepsilon_i} [z_0, .., z_{i-1}, z_i z_{i+1}, z_{i+2}, .., z_k] + (-1)^{(\deg z_k - 1)\varepsilon_{k-1}} [z_k z_0, .., z_{k-1}],$$

$$B([z_0, .., z_k]) = \sum_{i=0}^{k} (-1)^{(\varepsilon_{i-1}+1)(\varepsilon_k - \varepsilon_{i-1})} [1, z_i, .., z_k, z_0, .., z_{i-1}],$$

and $b_0(t) = b_1(t) = B(t) = 0$, where $\bar{A} = \oplus_{i>0} A^i$, $\deg[z_0, .., z_k] = \deg z_0 + \cdots + \deg z_k - k$, for $[z_0, .., z_k]$ in $\mathrm{N}(\mathcal{A})$, and $\varepsilon_i = \deg z_0 + \cdots + \deg z_i - i$. Since the formulas $b \circ B + B \circ b = 0$ and $b^2 = B^2 = 0$ hold, the derivation $b + t \cdot B$ is a differential with degree 1. Its homology is denoted $HC_*(\mathcal{A})$, which is called the cyclic homology of $\mathcal{A}$. We denote that of rational de Rham complex $(\Omega^*(X), \partial)$ of X by $HC^*(X; \mathbf{Q})$ and it call the *rational cyclic cohomology of* X. If $\mathcal{A}$ is formal, we see that $HC_*(\mathcal{A}) \cong HC_*(\mathcal{M}(\mathcal{A})) \cong HC_*(H_*(\mathcal{A}), 0)$ as algebras, which are induced from quasi-isomorphisms $\mathcal{A} \leftarrow \mathcal{M}(\mathcal{A}) \rightarrow H_*(\mathcal{A})$ and that $b = b_1$ in $\mathrm{N}(H_*(\mathcal{A}))$ since the differential of $H^*(\mathcal{A})$ is trivial.

Let $\mathcal{A} = (\wedge Z, d)$. Recall that $\mathcal{M}(ES^1 \times_{S^1} LX) \cong (\mathbf{Q}[t] \otimes \wedge\overline{Z} \otimes \wedge Z, D)$ in Section 2 of Chapter 0. There is the complex morphism from the cyclic bar complex to the CDGA,

$$\theta : (\mathbf{Q}[t] \otimes \mathrm{N}(\mathcal{A}), b + t \cdot B) \rightarrow (\mathbf{Q}[t] \otimes \wedge\overline{Z} \otimes \wedge Z, D)$$

given by $b \mapsto \delta$, $B \mapsto \beta$ and $\theta([z_0, .., z_k]) = (-1)^{\mu} z_0 \bar{z}_1 ... \bar{z}_k$, where $\mu = \deg z_1 + \deg z_3 + \deg z_5 + \cdots + \deg z_k$ if k is odd and $\mu = \deg z_1 + \deg z_3 + \deg z_5 + \cdots + \deg z_{k-1}$ if k is even. It is known that θ induces the isomorphism

$$\theta_* : HC^*(X; \mathbf{Q}) \cong H^*(ES^1 \times_{S^1} LX; \mathbf{Q}) \tag{3.1}$$

as $\mathbf{Q}[t]$-module ([**4**], [**6**], [**7**]). Then by (2.1) we can decompose $HC^*(X; \mathbf{Q})$ into a direct sum of a graded $\mathbf{Q}[t]$-module: $HC^*(X; \mathbf{Q}) \cong \mathbf{Q}[t] \oplus T^*$, where this T^* corresponds to T^* in $H^*(ES^1 \times_{S^1} LX; \mathbf{Q})$ by θ.

We start this section by calculating the rational cyclic cohomology of a bouquet of a finite number of odd dimensional spheres.

LEMMA 3.1. *If* $X_{(0)} \simeq \vee_{i=1}^{l} S^{2n_i+1}{}_{(0)}$, *then* $HC^*(X;\mathbf{Q})$ *is concentrated in even degrees.*

PROOF. Since X is formal, it is sufficient to show that $HC_*(H^*(X;\mathbf{Q}),0)$ is evenly graded. Let

$$H^*(X;\mathbf{Q}) = \wedge(x_1,..,x_l)/(x_ix_j;\ \ 1 \le i < j \le l),$$

where deg x_i is odd for any i. Then an element w with degree odd of $\mathbf{N}(H^*(X;\mathbf{Q}))[t]$ is uniquely expressed as

$$w = \sum_{n\ge 0} w_n \ \text{ with } w_n = \sum_i \left(\sum_{(i_0,..,i_n)\in I_{N_i}} a_{(i_0,..,i_n)}[x_{i_0},..,x_{i_n}] \right) \cdot t^{N_i},$$

where $a_{(i_0,..,i_n)} \in \mathbf{Q} - \{0\}$ for any $(i_0,..,i_n) \in I_{N_i}$ and $N_i \ne N_j$ if $i \ne j$.

Suppose that w is a $(b + t\cdot B)$-cocycle. Since Im $b \cap$ Im $t\cdot B\ = 0$ in this case, we have

$$B(w_n) = 0 \text{ for any } n.$$

The equation $B(w_0) = 0$ means $w_0 = 0$. Let $n > 0$. The cyclic group $\mathbf{Z}/(n+1)\mathbf{Z}$ action on $\overline{H^*(X;\mathbf{Q})}^{\otimes n+1}$ is given by letting its generator τ act by

$$\tau \cdot [x_{i_0},..,x_{i_{n-1}},x_{i_n}] = [x_{i_n},x_{i_0},..,x_{i_{n-1}}],$$

which is equivalent to the action by $\tau\cdot(i_0,..,i_{n-1},i_n) = (i_n,i_0,..,i_{n-1})$ on $(\mathbf{Z}/(l+1)\mathbf{Z} - \{0\})^{\times n+1}$. Then the index sets $\{I_{N_i}\}_i$ of w_n are represented as the following disjoint unions:

$$I_{N_i} = \cup_\gamma I^\gamma_{N_i}\ \ ;\ \ I^\gamma_{N_i} = \{\kappa \in I_{N_i} \mid \kappa = \tau^j\cdot\gamma \ \text{ for some } 0 \le j \le n\}.$$

For any index γ of $I^\gamma_{N_i}$, we can fix $k = k_\gamma$ and $0 = j_0 < j_1 < \cdots < j_k \le n$ such that

$$I^\gamma_{N_i} = \{\gamma = (i_0,..,i_n), \tau^{j_1}\cdot\gamma = (i_{n-j_1+1},..,i_n,i_0,..,i_{n-j_1}),\cdots,\tau^{j_k}\cdot\gamma = (i_{n-j_k+1},..,i_{n-j_k})\}.$$

Put $[x]_\gamma = [x_{i_0},..,x_{i_n}]$. Then we remark $\tau^{j_s}\cdot[x]_\gamma = [x]_{t^{j_s}\cdot\gamma} = [x_{i_{n-j_1+1}},..,x_{i_n},x_{i_0},..,x_{i_{n-j_1}}]$. The equation $0 = B(w_n) = B(\sum_i(\sum_{\gamma\in I_{N_i}}\sum_{s=0}^k a_{\gamma_s}\tau^{j_s}\cdot[x]_\gamma)\cdot t^{N_i}) = \sum_i(\sum_{\gamma\in I_{N_i}}\sum_{s=0}^k a_{\gamma_s}B(\tau^{j_s}\cdot[x]_\gamma))\cdot t^{N_i}$ means that

$$\sum_{s=0}^k a_{\gamma_s} = 0 \text{ for any } \gamma \in I_{N_i} \text{ and } i$$

since $B(\tau^j\cdot[x]_\gamma) = B([x]_\gamma)$ for any $0 \le j \le n$ and $\{B([x]_\gamma)\}_\gamma$ are linearly independent. If we put

$$[x]^a_b = [1,x_{i_{n-j_b+2}},..,x_{i_n},x_{i_0},..,x_{i_{n-j_b+1}}] + [1,x_{i_{n-j_b+3}},..,x_{i_{n-j_b+2}}] + \cdots + [1,x_{i_{n-j_a+1}},..,x_{i_{n-j_a}}]$$

for $0 \le a < b \le k$ and

$$[x]^k_0 = [1,x_{i_1},..,x_{i_n},x_{i_0}] + [1,x_{i_2},..,x_{i_1}] + \cdots + [1,x_{i_{n-j_k+1}},..,x_{i_{n-j_k}}],$$

then we have

$$b([x]^a_b) = [x_{i_{n-j_b+1}},..,x_{i_{n-j_b}}] - [x_{i_{n-j_a+1}},..,x_{i_{n-j_a}}] = \tau^{j_b}\cdot[x]_\gamma - \tau^{j_a}\cdot[x]_\gamma \text{ and}$$

$$b([x]^k_0) = [x_{i_0},..,x_{i_n}] - [x_{i_{n-j_k+1}},..,x_{i_{n-j_k}}] = [x]_\gamma - \tau^{j_k}\cdot[x]_\gamma,$$

respectively. From these equations, if we put

$$\begin{aligned}
u^\gamma_n = {} & (ka_{\gamma_0} +\ 0\ + a_{\gamma_2} + 2a_{\gamma_3} + \cdots + (k-1)a_{\gamma_k})[x]^0_1 \\
& + ((k-1)a_{\gamma_0} + ka_{\gamma_1} +\ 0\ + a_{\gamma_3} + \cdots + (k-2)a_{\gamma_k})[x]^1_2 \\
& + ((k-2)a_{\gamma_0} + (k-1)a_{\gamma_1} + ka_{\gamma_2} +\ 0\ + \cdots + (k-3)a_{\gamma_k})[x]^2_3 \\
& \vdots \\
& + (a_{\gamma_0} + 2a_{\gamma_1} + 3a_{\gamma_2} + \cdots + ka_{\gamma_{k-1}} +\ 0\ )[x]^{k-1}_k \\
& + (\ 0\ + a_{\gamma_1} + 2a_{\gamma_2} + 3a_{\gamma_3} + \cdots + ka_{\gamma_k})[x]^k_0,
\end{aligned}$$

then we have its image by differential b as

$$\begin{aligned}
b(u_n^\gamma) = & (ka_{\gamma_0} + 0 + a_{\gamma_2} + 2a_{\gamma_3} + \cdots)\tau^{j_1}[x]_\gamma \\
& - (ka_{\gamma_0} + 0 + a_{\gamma_2} + 2a_{\gamma_3} + \cdots)[x]_\gamma \\
& + ((k-1)a_{\gamma_0} + ka_{\gamma_1} + 0 + a_{\gamma_3} + \cdots)\tau^{j_2}[x]_\gamma \\
& - ((k-1)a_{\gamma_0} + ka_{\gamma_1} + 0 + a_{\gamma_3} + \cdots)\tau^{j_1}[x]_\gamma \\
& + ((k-2)a_{\gamma_0} + (k-1)a_{\gamma_1} + ka_{\gamma_2} + \cdots)\tau^{j_3}[x]_\gamma \\
& - ((k-2)a_{\gamma_0} + (k-1)a_{\gamma_1} + ka_{\gamma_2} + \cdots)\tau^{j_2}[x]_\gamma \\
& \vdots \\
& + (a_{\gamma_0} + 2a_{\gamma_1} \cdots + ka_{\gamma_{k-1}} + 0)\tau^{j_k}[x]_\gamma \\
& - (a_{\gamma_0} + 2a_{\gamma_1} \cdots + ka_{\gamma_{k-1}} + 0)\tau^{j_{k-1}}[x]_\gamma \\
& + (0 + a_{\gamma_1} + 2a_{\gamma_2} + 3a_{\gamma_3} + \cdots)[x]_\gamma \\
& - (0 + a_{\gamma_1} + 2a_{\gamma_2} + 3a_{\gamma_3} + \cdots)\tau^{j_k}[x]_\gamma \\
= & -(k+1)(a_{\gamma_0}[x]_\gamma + a_{\gamma_1}\tau^{j_1}\cdot[x]_\gamma + \cdots a_{\gamma_k}\tau^{j_k}\cdot[x]_\gamma) \\
= & -(k+1)\sum_{s=0}^{k}(a_{\gamma_s}\tau^{j_s}\cdot[x]_\gamma).
\end{aligned}$$

Thus we have

$$b\left(\sum_{n>0}\sum_i(\sum_{\gamma\in I_{N_i}}\frac{-1}{k_\gamma+1}u_n^\gamma)\cdot t^{N_i}\right) = \sum_{n>0}\sum_i(\sum_{\gamma\in I_{N_i}}\sum_{s=0}^{k_\gamma}a_{\gamma_s}\tau^{j_s}\cdot[x]_\gamma)\cdot t^{N_i} = \sum_{n>0}w_n = w,$$

that is, w is b-exact. Further we see that w is $(b+t\cdot B)$-exact since $B(u_n^\gamma) = 0$ for any $n > 0$ and γ. Hence there is no element with degree odd in $HC^*(H^*(X;\mathbf{Q}))$. □

LEMMA 3.2. *If $T^* = T^{even}$, then $H^*(X;\mathbf{Q}) = H^{odd}(X;\mathbf{Q})$.*

PROOF. Suppose $C^{even} \neq 0$. Then for a non zero element x of C^{even}, the element $\overline{x}$ is a generator of T^* of odd degree. This is a contradiction. So we have

$$C^{even} = 0.$$

Also suppose that w is a decomposable d-cocycle of $\wedge Z$ of even degree. From (2.1) of Chapter 0, $\beta(w) \neq 0$. Then we see that $\beta(w)$ is a δ-cocycle of odd degree. Since $T^* =$

T^{even}, there is an element v of $\wedge\overline{Z}\otimes\wedge Z$ such that $(\delta\circ\beta)(v) = \beta(w)$ from (2.2), that is, $d(-v) = w$ from (2.1) of Chapter 0. So we have $w = 0$ in $H^*(X;\mathbf{Q})$. Thus we see that $H^*(X;\mathbf{Q}) = H^{odd}(X)$. □

THEOREM 3.3. *T^* is concentrated in even degrees if and only if X has the rational homotopy type of a bouquet of a finite number of odd dimensional spheres.*

PROOF OF THEOREM 3.3. The "if" part follows from Lemma 3.1. The "only if" part follows by combining Lemma 3.2 and Theorem 1.5 of [**21**]. □

LEMMA 3.4. 1. *If $H^*(X;\mathbf{Q}) \cong \mathbf{Q}[x_{2n}]$ and $\deg x_{2n} = 2n$, then*

$\dim_{\mathbf{Q}} T^s = 1$ *for* $s = 2n(i+1) - 1$ *with* $i \geq 0$ *and*

$\dim_{\mathbf{Q}} T^s = 0$ *for other* s.

2. *If $H^*(X;\mathbf{Q}) \cong \mathbf{Q}[x_{2n}]/(x_{2n}^k)$ and $k > 1$, then*

$\dim_{\mathbf{Q}} T^s = 1$ *for* $s = 2n(jk+i+1) - 2j - 1$ *with* $j \geq 0,\ 0 \leq i \leq k-2$ *and*

$\dim_{\mathbf{Q}} T^s = 0$ *for other* s.

PROOF. Let $\mathbf{Q}\{S\}$ be the vector space generated by S. (1) is showed as follows: If $H^*(X;\mathbf{Q}) \cong \mathbf{Q}[x_{2n}]$, we have $\mathcal{M}(X) \cong (\wedge(x), 0)$ with $\deg x = 2n$. Then we see

$$T^* \cong \mathbf{Q}\{\overline{x}x^i;\ i \geq 0\}$$

as vector spaces. Note that $\deg(\overline{x}x^i) = 2n(i+1) - 1$.

(2) is showed as follows: If $H^*(X;\mathbf{Q}) \cong \mathbf{Q}[x_{2n}]/(x_{2n}^k)$, we have $\mathcal{M}(X) \cong (\wedge(x,y), d)$ with $\deg x = 2n$, $\deg y = 2kn - 1$, $d(x) = 0$, and $d(y) = x^k$. Then, following [**7**],

$$T^* \cong \mathbf{Q}\{\overline{y}^j\overline{x}x^i;\ j \geq 0,\ 0 \leq i \leq k-2\}$$

as vector spaces. Note that $\deg(\overline{y}^j\overline{x}x^i) = 2n(jk+i+1) - 2j - 1$. □

LEMMA 3.5. *If $T^* = T^{odd}$, then the algebra $H^*(X;\mathbf{Q})$ is generated by a single element of even degree.*

PROOF. Suppose $C^{odd} \neq 0$. Then for a non zero element x of C^{odd}, the element $\overline{x}$ is a generator of T^* of even degree. So we have

$$C^{odd} = 0.$$

Suppose $\dim_{\mathbf{Q}} C^{even} \geq 2$. If a basis of C^{even} is given as $\{x_1, \cdots, x_n\}$, then any $\overline{x}_i\overline{x}_j$ for $i \neq j$ is also a generator of T^* of even degree by (2.5). So we have

$$\dim_{\mathbf{Q}} C^{even} = 1.$$

If $\dim_{\mathbf{Q}} K > 1$, we can choose two elements y, z in K such that $d(y) = x^m$ and $d(z) = x^n$ with $2 \leq m \leq n$ for the only generator x of C and that they are linearly independent. Here we can assume $m < n$ from the construction of a minimal model in Section 1. Then we have

$$\mathrm{Ker}[d|_Z] \supset \mathbf{Q}\{x,\ z\} \underset{\neq}{\supset} \mathbf{Q}\{x\} = C = \mathrm{Ker}(d|_Z)$$

since $z \in \mathrm{Ker}[d|_Z]$ by $d(z) = x^n = d(x^{n-m} \cdot y)$. This contradicts the normality (see Definition 1.1). So we have

$$\dim_{\mathbf{Q}} K \leq 1.$$

Thus we see that $(\wedge Z, d) = (\wedge(x), 0)$ or $(\wedge Z, d) = (\wedge(x, y), d)$ with $d(x) = 0$ and $d(y) = x^n$ for some n. Since every non-zero element of $I(y)$ is not a d-cocycle, $(\wedge Z, d)$ is formal from Lemma 1.3. Thus we have this lemma from (1.2). □

THEOREM 3.6. *T^* is concentrated in odd degrees if and only if X has the rational homotopy type of the James' reduced product space $(S^{2m})_\infty$ of S^{2m} for some m or the $2mn$ skeleton $(S^{2m})_n$ of it for some n.*

PROOF OF THEOREM 3.6. Let $(S^{2m})_\infty$ be the James' reduced product space of S^{2m} for some m and $(S^{2m})_n$ the $2mn$ skeleton of it for some n. Then we recall that

$$H^*(X; \mathbf{Q}) \cong \mathbf{Q}[x]/(x^{n+1}) \text{ if and only if } X_{(0)} \simeq (S^{2m})_{n(0)} \text{ with } \deg x = 2m \tag{3.2}$$

as a special case of Corollary 2.6 of [**33**]. The "if" part follows from (3.2) and Lemma 3.4. The "only if" part follows from Lemma 3.5 and (3.2). □

The Gysin sequence ([**30**, p.419]) of S^1-fibration $S^1 \to LX \to ES^1 \times_{S^1} LX$:

$$\cdots \to H^{*+1}(LX; \mathbf{Q}) \to H^*(ES^1 \times_{S^1} LX; \mathbf{Q}) \xrightarrow{t\cdot} H^{*+2}(ES^1 \times_{S^1} LX; \mathbf{Q}) \xrightarrow{p^*} H^{*+2}(LX; \mathbf{Q}) \to \cdots$$

is equivalent to the Connes' periodic exact sequence ([**30**, p.60]):

$$\cdots \to HH^{*+1}(X; \mathbf{Q}) \to HC^*(X; \mathbf{Q}) \xrightarrow{S} HC^{*+2}(X; \mathbf{Q}) \to HH^{*+2}(X; \mathbf{Q}) \to \cdots$$

from (2) of Chapter 1 and (3.1), and the *Connes' periodic map S* corresponds the multiplication by t ([**6**, Theorem 2.4]).

M.Vigué-Poirrier proves in [**52**, Theorem B] that if X is formal, then the structure of $\mathbf{Q}[t]$-module on T^* is trivial, that is, $t \cdot T^* = 0$. Since a bouquet of a finite number of odd dimensional spheres, the James' reduced product space of S^{2m} for some m, and the $2mn$ skeleton of it for some n are formal ([**21**, Lemma 1.6] and [**13**, p.576], respectively), we obtain from Theorems 3.3 and 3.6:

COROLLARY 3.7. *If T^* is concentrated in even degrees or odd degrees, then $t \cdot T^* = 0$, i.e., $S(T^*) = 0$.*

This example indicates that the triviality can be induced not only by the formality of X or the algebra structure of $H^*(X; \mathbf{Q})$, but also by the character of a graded structure of the cyclic cohomology $HC^*(X; \mathbf{Q})$.

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