# The Study of Investment Decision

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## Introduction

This article is an attempt to solve the problem of optimal investment decision over time. In the first section we will discuss the principles laid down by J. Hirshleifer justly famous work about the optimal investment decision (1), and see the concept of time preference. The next problem is the one to maximize the profit in the situation of some investment opportunities for multiperiod.

In this case, there are three methods of optimizing the investment decision. The first is the constrained-maximization, the second is the constrained minimization, and the third is the method of maximization in regard to all the inputs and outputs as variables. The first and the second solution will lead to the first-order conditions which are independent of the interest rate.

But the third solution will lead to the first-order conditions which are dependent of the interest rate from the t th marketing date to the $\tau$ th marketing date. These methods of multiperiod profit maximization must be assumed to be multiperiod production function, but this production function is constructed subject to nonrealistic assumptions, to say that in the first and the second methods, the inputs during the t th period are finished to produced, and all the products are selled entirely on the t+1 th marketing date.

To improve the disadvantage of this formulation, we will introduce the investment opportunities function during the multiperiod.

By using this formulation, we can solve the general problem of internal rate of return of each investments, Finally we will discuss the problem of investment decision theory with risky prospect and its conditions to optimize.

### 1. Consumption decision

In order to establish the general investment decision, let us first review J. Hirshleifer's problem was to solve consumption decision between the actual income and the next period income by using the concept of time preference, we can formulate the present value(2).

$$V = y_0 + \frac{y_1}{1+i} \quad \cdots\cdots(1)$$

$y_0$ : income on period zero

$y_1$ : income on period one

$i$ : interest rate

$V$ : the total present value

In the meanwhile, consumption opportunities function is the next.

$$F(y_0,\ y_1) = 0 \quad \cdots\cdots(2)$$

We can formulate the objective function by using lagrange multiplier.

$$V = y_0 + y_1(1+i)^{-1} + \lambda F(y_0,\ y_1) \quad \cdots\cdots(3)$$

The first-order condition to maximize V are the partial derivatives of $y_0$, $y_1$ and $\lambda$

$$\partial V/\partial y_0 = 1 + \partial F/\partial y_0 = 0 \quad \cdots\cdots(4)$$

$$\partial V/\partial y_1 = (1+i)^{-1} + \lambda \partial F/\partial y_1 = 0$$

$$\partial V/\partial \lambda = F(y_0,\ y_1) = 0$$

and

$$i = -\,\partial y_1/\partial y_0 - 1 \quad \cdots\cdots(5)$$

If the rightpart is assumed the marginal internal rate of return from investment on the zero th marketing date, and the investment expenditure must be decided on the point to equal to the market interest rate.

The next problem is that of utility maximizing, assuming that two period incomes are the actual $y_0$ and the next $y_1$,

$$V_0 = y_0 + y_1(1+i)^{-1} \quad \cdots\cdots (6)$$

utility function to maximize

$$U = U(y_0, y_1) \quad \cdots\cdots (7)$$

and so

$$U = U(y_0, y_1) + (V_0 - y_0 - y_1(1+i)^{-1}) \quad \cdots\cdots (8)$$

and set its partial derivatives equal to zero.

$$\partial U/\partial y_0 = U_{y0} - \lambda = 0$$
$$\partial U/\partial y_1 = U_{y1} - \lambda(1+i)^{-1} = 0 \quad \cdots\cdots (9)$$

Eliminating $\lambda$ from the two equations of (9)

$$U_{y1} - U_{y0}(1+i)^{-1} = 0 \quad \cdots\cdots (10)$$

and so

$$i = U_{y1}/U_{y0} - 1$$

This implies that market interest rate equal to the marginal rate of time preference.

The second-order conditions to maximize the objective function are

$$\partial^2U/\partial y_0^2 < 0, \quad \partial^2U/\partial y_1^2 < 0 \quad \cdots\cdots (11)$$

To the next, we will generalize this consumption-expenditure decision problem. The present value of income is

$$V = \sum_{t=1}^{T}(1+\xi_{1t})^{-1}y_t \quad \cdots\cdots (12)$$

But $\xi_{1t} = (1+i)^{t-1} - 1$

and consumption opportunities function.

$$F(y_1 \cdots\cdots y_t) \quad \cdots\cdots (13)$$

$y_t$ : income on t period

By lagrange multiplier, we can formulate the objective function.

$$V = \sum_{t=1}^{T}(1+\xi_{1t})^{-1}y_t + \lambda F(y_1 \cdots\cdots y_t) \quad \cdots\cdots (14)$$

and set the partial derivatives equal to zero.

$$\partial V/\partial y_1 = (1+\xi_{1t})^{-1} + \lambda F_{y1} = 0$$
$$\partial V/\partial y_2 = (1+\xi_{1t})^{-1} + \lambda F_{y2} = 0$$
$$\partial V/\partial y_t = (1+\xi_{1t})^{-1} + \lambda F_{yt} = 0$$
$$\partial V/\partial \lambda = F(y_1 \cdots\cdots y_t) = 0 \quad \cdots\cdots (15)$$
$$(t = 1, \cdots\cdots, T)$$

and

$$(1+\xi_{kt})^{-1} = F_{yt}/F_{yk} \quad \cdots\cdots (16)$$

That is, time preference of consumption from k th marketing date to the t th marketing date denoted by $\xi_{kt}$.

$$\xi_{kt} = -\partial y_t/\partial y_k - 1 \quad \cdots\cdots (17)$$

We can get the marginal internal rate of return from investment expenditure on the k th marketing date.

## 2. Profit maximization

In generally the constrained-minimization problem are solved on the assumtions that are given all the inputs and outputs except the input on the t th marketing date. In that case, the first-order conditions are equal to the famous static conditions to maximize profit(3).

$$-\partial g_{kt}/\partial g_{jt} = P_{kt}/P_{jt} \cdots\cdots\cdots (18)$$

$$(j,\ k = s+1, \cdots\cdots, m)$$

That is, TRS is equal to the rate of factor price.
In this case, production function are formulate as the next (3)

$$\begin{pmatrix} 0 \cdots\cdots\cdots 0 & g^0_{s+1\cdot 1} \cdots\cdots g^0_{m\cdot 1} \\ g^0_{1\cdot 2} \cdots\cdots\cdots g^0_{s\cdot 2} & g^0_{s+1\cdot 2} \cdots\cdots g^0_{m\cdot 2} \\ g^0_{1\cdot 3} \cdots\cdots\cdots g^0_{s\cdot 3} & g^0_{s+1\cdot 3} \cdots\cdots g^0_{m\cdot 3} \\ - \quad\quad - & - \quad\quad - \\ - \quad\quad - & - \quad\quad - \\ g^0_{1\cdot t} \cdots\cdots\cdots g^0_{s\cdot t} & g_{s+1\cdot t} \cdots\cdots g_{m\cdot t} \\ - \quad\quad - & - \quad\quad - \\ - \quad\quad - & - \quad\quad - \\ - \quad\quad - & g^0_{s+1\cdot L} \cdots\cdots g^0_{m\cdot L} \\ g^0_{1\cdot t+1} \cdots\cdots g^0_{s\cdot L+1} & 0 \cdots\cdots\cdots 0 \end{pmatrix} = F \cdots\cdots\cdots (19)$$

But the constrained-maximization can be formulate, assuming that only the outputs on the t th marketing date are variable, and all the other outputs and inputs are fixed. In this case, multiperiod production function is

$$\begin{pmatrix} 0 \cdots\cdots\cdots 0 & g^0_{s+1\cdot 1} \cdots\cdots g^0_{m\cdot 1} \\ g^0_{1\cdot 2} \cdots\cdots\cdots g^0_{s\cdot 2} & g^0_{s+1\cdot 2} \cdots\cdots g^0_{m\cdot 2} \\ - \quad\quad - & - \quad\quad - \\ - \quad\quad - & - \quad\quad - \\ g_{1\cdot t} \cdots\cdots\cdots g_{s\cdot t} & g^0_{s+1\cdot t} \cdots\cdots g^0_{m\cdot t} \\ - \quad\quad - & - \quad\quad - \\ - \quad\quad - & g^0_{s+1\cdot L} \cdots\cdots g_{m\cdot L} \\ g^0_{1\cdot L+1} \cdots\cdots g^0_{s\cdot L+1} & 0 \cdots\cdots\cdots 0 \end{pmatrix} = F \cdots\cdots\cdots (20)$$

$$(j,\ k=1, \cdots\cdots, s)$$

The objective function to maximize profit subject to the above function.

$$\pi = \sum_{t=1}^{L+1} \sum_{j=1}^{m} P_{jt} g_{jt} (1 + \xi_{1t})^{-1} + \lambda F(\cdots\cdots) \cdots\cdots\cdots (21)$$

That is, the optimal output set can be decided by the partial derivatives to equal to zero. The optimal RPT for the outputs at the given marketing date are independent to the interest rate. But these constrained-optimization can not be compared with the marginal internal rate of return for the investments between the two different periods.

Then, assuming that all inputs and outputs are variable, we can compare with each marginal rate of return from each input.
The objective function is

$$V = \sum_{j=2}^{s} P_{jt} g_{jt} (1 + \xi_{1t})^{-1} - \sum_{j=s+1}^{m} P_{jt} g_{jt} (1 + \xi_{1t})^{-1} + \lambda F(g_{12}, \cdots\cdots, g_{mL}) \cdots\cdots (22)$$

The $g_{jt}$ are the outputs $(j = 2, \cdots\cdots s)$, but the $g_{js}(j = s+1, \cdots\cdots m)$ are the inputs. The first-order condion for maximization of profit can be derived by each output and input on the different periods.

$$\partial V/\partial g_{jt} = P_{jt}(1 + \xi_{1t})^{-1} + \lambda F^1_{jt} = 0$$

$$\partial V/\partial g_{js} = P_{js}(1 + \xi_{1s})^{-1} + \lambda F^1_{js} = 0$$

$$\partial V/\partial \lambda = F(g_{12}, \cdots\cdots, g_{mL}) \cdots\cdots\cdots (23)$$

Eliminating $\lambda$ from (23)

$$-P_{js}(1 + \xi_{1\cdot s})^{-1}/P_{jt}(1 + \xi_{1\cdot t})^{-1} = F_{js}/F_{jt}$$

That is, $(s > t)$

$$-P_{js}(1+\xi_{ts})^{-1}/P_{jt} = \partial g_{jt}/\partial g_{js} \cdots\cdots (24)$$

We can understand these equations by the three meanings. At the first, when all variables $g_{jt}$ and $g_{js}$ are the inputs, (24) mean the rate of technical substitution between the two periods in the future and the rate of techical substitution equal to the rate of the factor price, discounted by the interest rate from the t th marketing date to the s th marketing date. $(t, s=1, \cdots\cdots L)$ $(j, k=s+1, \cdots\cdots m)$
To the next, we can understand (24) equations, as the rate of products transformation of the k goods and the j goods.
The (24) equations mean that the rate of product transformation equal to the rate of the price of the k product at the t th marketing date and the price of the j product at the s th marketing date, discounted by the interest rate from t to s period.
The next problem is the effect of the input at the t th marketing date to the output at the s th marketing date. This formulation is

$$\partial g_{js}/\partial g_{kt} = P_{kt}/P_{js}(1+\xi_{ts})^{-1} \cdots\cdots (25)$$

$$(t = 1, \cdots\cdots, L)$$
$$(s = 2, \cdots\cdots L+1)$$
$$(k = s+1, \cdots\cdots, m)$$
$$(j = 1, \cdots\cdots, s)$$

This rate of marginal productivity equal to the rate of the price of product at the t th period and the factor price at the s th period, discounted by the interest rate from t to s periods.

This formulation must be assumed that the multiperiod production function can get definitely. But we can construct briefly by substituting the multiperiod production function to the investment-revenue function. The investment-revenue function must be assumed that the fixed production set and price set exist as variables already.

$$H(I_{j1}, \cdots\cdots I_{jL}, R_{j2} \cdots\cdots, R_{j \cdot L+1}) = 0$$

so that we briefly construct the objective function to maximize profit

$$\pi^* = \sum\sum R_{j \cdot t}(1+\xi_{1t})^{-1} - \sum\sum I_{jt}(1+\xi_{1t})^{-1} + \mu \sum H_j(I_{j1} \cdots\cdots I_{jL}, R_{j1} \cdots\cdots R_{j \cdot L+1}) \cdots\cdots (26)$$

In this case, the marginal rate of return from investment can be understand in two cases, one is the marginal return at the t th marketing date from the investment at the t th marketing date, the two is the marginal return at the t th marketing date from the investment at the $\tau$ th marketing date. The first can be formulate as the next equation.

$$(1+\xi_{1t})^{-1}(1+\partial I_{jt}/\partial R_{jt}) = 0$$

But the second can be formulate from the first-order condition for profit maximization.

$$\partial \pi^*/\partial R_\tau = (1+\xi_1)^{-1} + \lambda \partial H/\partial R_\tau = 0$$
$$\partial \pi^*/\partial I_t = -(1+\xi_{1t})^{-1} + \lambda \partial H/\partial I_t = 0$$
$$\partial \pi^*/\partial \mu = H_j(I_{j1} \cdots\cdots I_{jL}, R_{j1} \cdots\cdots R_{jL+1}) = 0 \cdots\cdots (27)$$

Eliminating $\lambda$ from (27) equations.

$$(1+\xi_{1t})^{-1}((1+\xi_{t\tau})^{-1} + \partial I_t/\partial R_\tau)) = 0$$

This equation mean that the marginal rate of return for investment at the same period equal to zero and at the different period equal to $\xi_{t\tau}$. We can understand clearly the difference between the production function and the investment-revenue function.

### 3. Multiperiod investment with risky prospect,

There are many articles about the investment plan with risky situation and can not state about these studies in this article, but these studies was about the plan with the single production period or consumption period. We can consider the two cases with multiperiod investment with risky prospect. One is the case independent

with the profits at the different periods. The second is the case dependent with the profits at the t th marketing date and at the $\tau$ th marketing date.
The first situation can show by the next equation.

$$\sum_{t=1}^{T} K_t^{t-1}\sigma_t\sigma_w = 0 \quad (t \neq w)\sigma_t\sigma_w : \text{covariance between } R_t \text{ and } R_w$$

In this case, we can formulate the objective function as the problem of profit maximization.

$$\pi = \sum_{t=2}^{L+1} F(R_t, \sigma_t)\ (1+\xi_{1t})^{-1} - \sum_{t=1}^{L} I_t\ (1+\xi_{1t})^{-1} + \mu H\,(I_1, \cdots I_L,\ R_2 \cdots R_{L+1},\ \sigma_2 \cdots \sigma_{L+1}) \cdots \qquad (28)$$

Setting the partial derivatives by each variables,

$$\partial\pi/\partial R_t = \sum_{t=2}^{L+1} F^1_{Rt}\,(R_t,\ \sigma_t)\ (1+\xi_{1t})^{-1} + \mu H^1_{Rt} = 0$$

$$\partial\pi/\partial\sigma_t = \sum_{t=2}^{L+1} F^1_{\sigma t}(R_t,\ \sigma_t)\ (1+\xi_{1t})^{-1} + \mu H^1_{\sigma t} = 0$$

$$\partial\pi/\partial I_t = (1+\xi_{1t})^{-1} + \mu H^1_{It} = 0$$

$$\partial\pi/\partial\mu = H\ (I_1 \cdots\cdots I_L\ R_2 \cdots\cdots R_{L+1},\ \sigma_2 \cdots\cdots \sigma_{L+1}) = 0 \qquad (29)$$

And the second-order condition for the profit-maximization.

$$\partial^2\pi/\partial R^2 = \sum_{t=2}^{L+1} F^{11}_{Rt}(R_t,\ \sigma_t)\,(1+\xi_{1t})^{-1} + \mu H^{11}_{Rt} < 0$$

$$\partial^2\pi/\partial\sigma_t^2 = \sum_{t=2}^{L+1} F^{11}_{\sigma t}(R_t, \sigma_t)\ (1+\xi_{1t})^{-1} + \mu H^{11}_{\sigma t} < 0$$

$$\partial^2\pi/\partial I_t^2 = H^{11}_{It}(I_1 \cdots\cdots I_L,\ R_2 \cdots\cdots R_{L+1},\ \sigma_2 \cdots\cdots \sigma_{L+1}) < 0 \qquad (30)$$

However, more realistically there are the relations between the profits at the different times. It is very difficult in the case that there exist variance-covariance relationship beween the profits at the different times.
That is, many assumptions must be settled regard to the objective function and setting the expected value.

The present value is assumed to be the additive function(6).

$$E\ (PV) = \sum_{t=2}^{T} K_t^{t-1}\mu_t \qquad (31)$$

$\mu_t$ : the expected income
k : discounted rate $r = (1-k_t)/k_t$

In this case, variance of the present value,

$$V\ (PV) = \sum (K_t^{t-1}\sigma_t)^2 + \sum_{t \neq w}\sum K_t^{t-1}K_w^{w-1}\rho\sigma_t\sigma_w (t,\ w = 1 \cdots\cdots\cdots m)$$

$\rho$ is the correlation coefficient between $\mu_t$ and $\mu_w$
We can set the problem of minimizing the variance V(PV). In this case the solutions are choiced only the securiety, and the present value is beyond question. If the objective function assumed to be the next relation,

$$U(PV) = X + bX \qquad (X = \sum_{t=1}^{n} K_t^{t-1}\mu_t)$$

Then we can get the expected utility function

$$EU(PV) = E(PV) + b(E(PV))^2 + bV(PV) \qquad (32)$$

setting the partial derivatives from equation (32)

$$EU(PV)/\partial\mu_t = K_t^{t-1} + 2b\sum_{j=1}^{n} K_t^{t-1}K_j^{j-1}\mu + b\partial V(PV)/\partial\mu_t = 0$$

$$\partial^2 EU(PV)/\partial\mu_t^2 = 2b\sum_{j=1}^{n} K_t^{t-1}K_j^{j-1}\partial\mu_j/\partial\mu_t + b\partial^2 V(PV)/\partial\mu < 0 \qquad (33)$$

As writting at the above sentence, we must set the many assumtions in this case, and need labourios culculations. I much owe to Professor Minoru, Fukuda for very helpful comments on this subjects.

## References

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# 投資決定の研究

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本稿は計画問題を次の三点から考察したものである.

(1) 将来所得配分の決定 J. Hirshleifer モデルを一般化して最適消費決定を行ない, 時間選好の関係を調べた.

(2) 多数期間の利潤最大化理論を条件付き最大化, 最小化問題として解いた場合は利子率に独立な条件が導びかれ, 1期間内の投資の限界内部収益率が導びかれるのに対して, 投資収益関数を用いた場合及び多数期間の全投資及び全生産量を変数とした場合の限界内部収益率は利子率が関係する. その関係を検討した.

(3) リスクを含む多数期間の投資問題は二点に関して論じている. 1点は異期間利潤が相関関係を有しない場合であり, 他の1つはそれらが互いに独立でない場合を見た. いずれの場合も期待効用に関する仮定が必要であり, 多くの困難があることが解る.